CM values and Fourier coefficients of harmonic Maass forms

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Zusammenfassung

In der vorliegenden Dissertation wird gezeigt, dass die Fourier-Koeffizienten gewisser harmonischer Maaß Formen halb-ganzen Gewichts die getwisteten Spuren von CM-Werten von harmonischen Maaß Formen ganzen Gewichts sind. Diese Ergebnisse verallgemeinern Arbeiten von Zagier, Bruinier, Funke und Ono über die Spuren von CM-Werten von harmonischen Maaß Formen von Gewicht 0 und -2.

Wir betrachten zwei Thetaliftungen, den sogenannten Kudla-Millson und den Bruinier-Funke Thetalift, um diese Resultate zu erhalten. Beide Liftungen haben interessante Anwendungen. Insbesondere kann mit Hilfe des Bruinier-Funke Lifts gezeigt werden, dass das Verschwinden der zentralen Ableitung der Hasse-Weil Zeta-Funktion einer elliptischen Kurve E über $\mathbb{Q}$ mit der Algebraizität der Spur von CM-Werten einer zu E assoziierten harmonischen Maaß Form zusammenhängt.

Contents

Introduction

In this thesis, we show that the Fourier coefficients of certain half-integral weight harmonic Maass forms are given as “twisted traces” of CM values of integral weight harmonic Maass forms. These results generalize work of Zagier, Bruinier, Funke, and Ono on traces of CM values of harmonic Maass forms of weight 0 and -2 [Zag02, BF04, BO13].

We utilize two theta lifts: one of them is a generalization of the Kudla-Millson theta lift considered in [BF04, BO13, AE13] and the other one is defined using a theta kernel recently studied by Hövel [Höv12].

Both of the lifts have interesting applications. For instance, we show that the vanishing of the central derivative of the Hasse-Weil zeta function of an elliptic curve E over $\mathbb{Q}$ is encoded by the Fourier coefficients of a harmonic Maass form arising from the Weierstrass ζ-function of E.

Parts of this thesis were published in [Alf14] and in a joint paper with Michael Griffin, Ken Ono, and Larry Rolen [AGOR].

Harmonic weak Maass forms

We first define the notion of harmonic weak Maass forms. The space of such forms was introduced by Bruinier and Funke in [BF04]. Here and in the following we let $z := x+iy \in \mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$, where $x, y \in \mathbb{R}$, and we let $q := e^{2\pi i z}$. For an integer $N \geq 1$ we have the congruence subgroup $\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N}\}$. We let $k \in \mathbb{Z}$.

A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}$ is called a *harmonic weak Maass form* of weight k for $\Gamma_0(N)$ if the following conditions are satisfied:

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.

(ii) $\Delta_k f = 0$, where $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ is the weight k Laplace operator.

(iii) There is a polynomial $P_f = \sum_{n \leq 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f(z) = O(e^{-\epsilon y})$, as $y \to \infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps.

We denote the space of such forms by $H_k^+(N)$. If $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ we require a slightly modified transformation behavior in (i). In the body of this thesis we will consider vector valued analogs of these spaces.

We let $k \neq 1$. A weight k harmonic Maass form[1] $f(z)$ has a Fourier expansion of the form

$$f(z) = f^+(z) + f^-(z) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n<0} c^-(n) \Gamma(1-k, 4\pi |n| y) q^n, \tag{0.1}$$

at the cusp ∞ and similar Fourier expansions at the other cusps. Here, $\Gamma(\alpha, x)$ is the incomplete Gamma-function. The function $f^+(z) = \sum_{n \gg -\infty} c^+(n) q^n$ is the *holomorphic part* of $f(z)$, and $f^-(z)$ the *non-holomorphic part*. If f^- is nonzero, then f^+ is also called a *mock modular form*. This is due to the connection between harmonic Maass forms and Ramanujan's mock theta functions: thanks to work of Zwegers [Zwe02] we know that every such mock theta function is the holomorphic part of a weight $1/2$ harmonic Maass form.

If f^- vanishes at the cusp ∞, then $f = f^+$ is a *weakly holomorphic modular form* of weight k. We denote the space of such forms by $M_k^!(N)$. The subspace of *modular forms* $M_k(N)$ of $M_k^!(N)$ consists of those functions that are holomorphic at all cusps and the subspace of *cusp forms* $S_k(N)$ consists of the forms that vanish at all cusps.

Bruinier and Funke [BF04] showed that harmonic Maass forms are intimately connected to cusp forms via the differential operator

$$\xi_k(f) := 2iy^k \overline{\frac{\partial}{\partial \bar{z}} f(z)}.$$

Every weight k cusp form is the image of infinitely many weight $2-k$ harmonic Maass forms under ξ_{2-k}. If the cusp form $\xi_k(f)$ carries arithmetic information it is an interesting question whether there are "canonical" preimages of $\xi_k(f)$ that also encode arithmetic information.

Throughout the introduction, we will use the variable $z = x + iy \in \mathbb{H}$ for integer weight forms and $\tau = u + iv \in \mathbb{H}$ for half-integer weight forms. We denote both $e^{2\pi i z}$ and $e^{2\pi i \tau}$ by q. It will be clear from the context whether $q = e^{2\pi i z}$ or $q = e^{2\pi i \tau}$.

Traces of singular moduli

A classical result states that the values of the modular j-invariant at quadratic irrationalities, called "singular moduli", are algebraic integers. Their properties have been intensively studied since the 19th century. In an influential paper [Zag02], Zagier showed that the (twisted) traces of these values occur as the Fourier coefficients of weakly holomorphic modular forms of weight $1/2$ and $3/2$.

To be more precise we let p be a prime or $p = 1$ and D be a negative integer congruent to a square modulo $4p$. We consider the set $\mathcal{Q}_{D,p}$ of positive definite integral binary quadratic forms $[a,b,c] = ax^2 + bxy + cy^2$ of discriminant $D = b^2 - 4ac$ such that c is divisible by p. If $p = 1$, we simply write $\mathcal{Q}_D$. For each form $Q = [a,b,c] \in \mathcal{Q}_{D,p}$ there is an associated CM point $\alpha_Q = \frac{-b+\sqrt{D}}{2a}$ in $\mathbb{H}$. These points are called CM points, since the associated elliptic curve has complex multiplication. The group $\Gamma_0(p)$ acts on $\mathcal{Q}_{D,p}$ with finitely many orbits.

[1] For convenience we shall refer to harmonic weak Maass forms as harmonic Maass forms.

Let $\Delta \in \mathbb{Z}$ be a positive fundamental discriminant (possibly 1) and d be a positive integer such that $-d$ and Δ are squares modulo $4p$. To ease the exposition we assume that $-d$ is a fundamental discriminant throughout the introduction.

For a weakly holomorphic modular form F of weight 0 for $\Gamma_0(p)$, we consider the modular trace function

$$\mathbf{t}_\Delta(F;d) = \frac{1}{\sqrt{\Delta}} \sum_{Q\in\Gamma_0(p)\backslash\mathcal{Q}_{-d\Delta,p}} \frac{\chi_\Delta(Q)}{|\overline{\Gamma_0(p)}_Q|} F(\alpha_Q), \tag{0.2}$$

where $\overline{\Gamma_0(p)}_Q$ denotes the stabilizer of Q in $\overline{\Gamma_0(p)}$, the image of $\Gamma_0(p)$ in $\mathrm{PSL}_2(\mathbb{Z})$. The function χ_Δ is a genus character, defined for $Q = [a,b,c] \in \mathcal{Q}_{-d\Delta,p}$ by

$$\chi_\Delta(Q) = \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } (a,b,c/p,\Delta) = 1,\ \Delta \mid (b^2-4ac), \text{ and } \frac{b^2-4ac}{\Delta} \text{ is a square mod } 4p, \\ 0, & \text{otherwise.} \end{cases}$$

Here, n is any integer prime to Δ and represented by one of the quadratic forms $[a,b,c]$ or $[pa,b,c/p]$. It is known that $\chi_\Delta(Q)$ is $\Gamma_0(p)$-invariant [GKZ87]. Note that for $\Delta = 1$ we have $\chi_\Delta(Q) = 1$ for all $Q \in \mathcal{Q}_{-d,p}$.

Let $J(z) = j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots$, $q := e^{2\pi i z}$, be the normalized Hauptmodul for the group $\mathrm{PSL}_2(\mathbb{Z})$. By the theory of complex multiplication it is known that $\mathbf{t}_\Delta(J;d)$ is a rational integer [Shi94, Section 5.4].

Zagier [Zag02, Theorem 6] proved that for $p = 1$ and $\Delta > 0$ the "generating series" of these traces,

$$g_\Delta(\tau) = q^{-\Delta} - \sum_{\substack{d\geq 0 \\ d\equiv 0,3(4)}} \mathbf{t}_\Delta(J;d)q^d,$$

is a weakly holomorphic modular form of weight 3/2 for $\Gamma_0(4)$. Here, we set $\mathbf{t}_\Delta(J;0) = 2$, if $\Delta = 1$ and $\mathbf{t}_\Delta(J;0) = 0$, otherwise. At the same time, these traces occur as the coefficients of weight 1/2 weakly holomorphic modular forms. We have that

$$f_d(\tau) = q^{-d} + \sum_{\substack{\Delta>0 \\ \Delta\equiv 0,1(4)}} \mathbf{t}_\Delta(J;d)q^\Delta$$

is a weakly holomorphic modular form of weight 1/2.

Twisted traces of CM values of harmonic Maass forms

Zagier's results were generalized in various directions, mostly for modular curves of genus zero [BO07, DJ08, Kim09, MP10]. Building upon previous work of Funke [Fun02], Bruinier and Funke [BF06] showed that Zagier's function g_1 can be obtained as a special case of a theta lift using a kernel function constructed by Kudla and Millson [KM86]. This lift, called *Kudla-Millson theta lift*, maps a harmonic weak Maass form F of weight 0 on a modular curve of arbitrary genus to a harmonic Maass form $\mathcal{I}^{\mathrm{KM}}(\tau,F)$ of weight 3/2. It is given by the following theta integral

$$\mathcal{I}^{\mathrm{KM}}(\tau, F) = \int_{\Gamma_0(N)\backslash\mathbb{H}} F(z)\Theta(\tau, z, \varphi_{\mathrm{KM}}).$$

Here, the theta series $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ is associated to a certain even lattice L of signature $(1, 2)$ and a certain Schwartz function φ_{KM} first defined in [KM86]. It has weight $3/2$ in the variable $\tau \in \mathbb{H}$ (for $\mathrm{SL}_2(\mathbb{Z})$ resp. a suitable generalization), is invariant under the action of $\Gamma_0(N) \subset \mathrm{SO}(1, 2)$ in the variable $z \in \mathbb{H}$, and is valued in the differential forms of Hodge type $(1, 1)$. The Fourier coefficients of positive index of $\mathcal{I}^{\mathrm{KM}}(\tau, F)$ are given by the traces of the CM values of F [BF06, Theorem 7.8].

Theta functions of this kind can be used to lift modular forms from one group to another. This phenomenon is explained by the fact that the two groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SO}(1, 2)$ form a reductive dual pair in the sense of Howe. Such theta lifts are a rich source of information on modular forms and their generalizations (for example in the work by Shimura, Shintani, Borcherds, Kudla, and Bruinier just to name a few).

The results of Bruinier and Funke were later generalized to the case $\Delta \neq 1$ by Ehlen and the author [AE13]. We developed a systematic approach to twist vector valued modular forms that transform with a certain Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. With this method and the results of Bruinier and Funke we studied the generating series of twisted traces of harmonic Maass forms and recovered Zagier's functions g_Δ as special cases of the twisted Kudla-Millson lift.

Recently, Bruinier and Ono [BO13] obtained a result similar to that of Bruinier and Funke for the coefficients of weight $-1/2$ harmonic Maass forms. Using the Maass raising and lowering operators they modified the Kudla-Millson lift such that it lifts from weight -2 to weight $-1/2$. In this way, they obtained a finite algebraic formula for the partition function $p(n)$ in terms of traces of the CM values of the derivative of a weakly holomorphic modular form F of weight -2 on $\Gamma_0(6)$.

In this thesis, we generalize the Kudla-Millson lift in two ways. Firstly, we extend the lift to other weights (as suggested by Bruinier and Ono) and secondly, we include twisted traces. Moreover, we show that there is another theta lift, the *Bruinier-Funke theta lift*, that generalizes Zagier's functions f_d in the same way as the Kudla-Millson lift generalizes the g_Δ's.

To make this more precise recall that p is a prime and that Δ is a positive fundamental discriminant. Moreover, $-d$ is a negative fundamental discriminant such that $-d$ and Δ are squares modulo $4p$. By $\mathcal{Q}_{-d\Delta,p}$ we denote the set of positive and negative definite integral binary quadratic forms $[a, b, c] = ax^2 + bxy + cy^2$ of discriminant $-d\Delta$ such that $c \equiv 0 \pmod{p}$. We assume that $(\Delta, 2p) = 1$ if $p \neq 1$.

Let $k \in \frac{1}{2}\mathbb{Z}$. We will modify the Kudla-Millson theta lift using the Maass raising and lowering operators R_k and L_k. These are differential operators that raise respectively lower the weight by 2. By $R_k^n := R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k$ we denote the iterated raising operator and by $L_k^n = L_{k-2(n-1)} \circ \cdots L_{k-2} \circ L_k$ the iterated lowering operator. Note that $R_{-2k}^k F$ has weight 0 for a harmonic Maass form F of weight $-2k < 0$ for $\Gamma_0(p)$ but does not inherit the analytic properties of F.

The generalization of the Kudla-Millson lift of weight $-2k<0$ harmonic Maass forms F is defined by

$$\mathcal{I}^{\mathrm{KM}}_{\Delta}(\tau,F)=R^{k/2}_{3/2,\tau}\int_{\Gamma_0(p)\backslash\mathbb{H}}(R^k_{-2k,z}F)(z)\Theta_\Delta(\tau,z,\varphi_{\mathrm{KM}}),\quad \text{for } k \text{ even},$$

and by

$$\mathcal{I}^{\mathrm{KM}}_{\Delta}(\tau,F)=L^{(k+1)/2}_{3/2,\tau}\int_{\Gamma_0(p)\backslash\mathbb{H}}(R^k_{-2k,z}F)(z)\Theta_\Delta(\tau,z,\varphi_{\mathrm{KM}}),\quad \text{for } k \text{ odd},$$

where $\Theta_\Delta(\tau,z,\varphi_{\mathrm{KM}})$ is a twisted version of $\Theta(\tau,z,\varphi_{\mathrm{KM}})$.

Moreover, we define the twisted modular trace function for $F\in H_{-2k}(p)$ by

$$\mathbf{t}_\Delta(F;d)=\sum_{Q\in\Gamma_0(p)\backslash\mathcal{Q}_{-d\Delta,p}}\frac{\chi_\Delta(Q)}{|\overline{\Gamma_0(p)}_Q|}R^k_{-2k}F(\alpha_Q).$$

We have the following theorem.

Theorem 1. *Let p,Δ,d be as above and let $F\in H^+_{-2k}(p)$ be a harmonic Maass form of negative weight $-2k$ for $\Gamma_0(p)$ that is invariant under the Fricke involution $z\mapsto -\frac{1}{pz}$.*

(i) If k is even, the Kudla-Millson lift of F is a weakly holomorphic modular form of weight $3/2+k$ for $\Gamma_0(4p)$. The d-th coefficient of the holomorphic part of the lift is given by

$$\left(-\frac{4\pi d}{|\Delta|}\right)^{k/2}\mathbf{t}_\Delta(F;d).$$

(ii) If k is odd, the Kudla-Millson lift of F is a harmonic Maass form of weight $1/2-k$ for $\Gamma_0(4p)$. The d-th coefficient of the holomorphic part of the lift is given by

$$\left(\frac{|\Delta|}{4\pi d}\right)^{(k+1)/2}\prod_{j=0}^{(k-1)/2}\left(\frac{k}{2}+j\right)\left(j-\frac{k+1}{2}\right)\mathbf{t}_\Delta(F;d).$$

The Kudla-Millson lift is weakly holomorphic if and only if the twisted L-function of $\xi_{-2k}(F)$ vanishes at $k+1$, that is

$$L(\xi_{-2k}(F),\Delta,k+1)=0.$$

In particular, this is the case when F is weakly holomorphic.

Remark 2. The theorem also gives an interesting new criterion on the nonvanishing of the twisted central L-values (see [Gol79, OS98, Ono01] for more information on this topic).

Remark 3. The theorem generalizes the functions g_Δ to higher weight as indicated in Section 9 of [Zag02].

We now explain a similar framework for the generalization of the weight 1/2 forms f_d via a theta lift. Recall that $-d$ is a negative fundamental discriminant. We use the Millson theta function $\Theta_{-d}(\tau, z, \psi_{\mathrm{KM}})$ as an integration kernel. The theta series $\Theta_{-d}(\tau, z, \psi_{\mathrm{KM}})$ is associated to a certain even lattice of signature $(1,2)$ and a certain Schwartz function ψ_{KM} first defined in [KM90]. It has weight 1/2 in the variable τ and is invariant under the action of $\Gamma_0(N)$ in the variable z.

The theta function $\Theta_{-d}(\tau, z, \psi_{\mathrm{KM}})$ was recently studied by Hövel in his PhD thesis [Höv12]. He lifted harmonic Maass forms of weight 1/2 to obtain locally harmonic Maass forms of weight 0, i.e. going to the direction "opposite" to ours.

For a harmonic Maass form F of weight $-2k < 0$ for $\Gamma_0(p)$ we define the Bruinier-Funke theta lift of F by

$$\mathcal{I}^{\mathrm{BF}}_{-d}(\tau, F) = L^{k/2}_{1/2,\tau} \int_{\Gamma_0(p)\backslash\mathbb{H}} (R^k_{-2k,z}F)(z)\Theta_{-d}(\tau, z, \psi_{\mathrm{KM}})d\mu(z), \quad \text{for } k \text{ even},$$
$$\mathcal{I}^{\mathrm{BF}}_{-d}(\tau, F) = R^{(k+1)/2}_{1/2,\tau} \int_{\Gamma_0(p)\backslash\mathbb{H}} (R^k_{-2k,z}F)(z)\Theta_{-d}(\tau, z, \psi_{\mathrm{KM}})d\mu(z), \quad \text{for } k \text{ odd},$$

where $d\mu(z) = \frac{dxdy}{y^2}$.

To describe the Fourier expansion of the holomorphic part we need to refine the definition of the modular trace function. Recall that $\Delta \neq 1$ is a positive fundamental discriminant and that Δ and $-d$ are squares modulo $4p$. We define two subsets $\mathcal{Q}^+_{-d\Delta,p}$ and $\mathcal{Q}^-_{-d\Delta,p}$ of $\mathcal{Q}_{-d\Delta,p}$ depending on the sign of a (where $Q = [a,b,c]$): for $a > 0$, the form is in $\mathcal{Q}^+_{-d\Delta,p}$ and for $a < 0$ it is contained in $\mathcal{Q}^-_{-d\Delta,p}$. We define the modular trace functions $\mathbf{t}^+_{-d}(f;\Delta)$ and $\mathbf{t}^-_{-d}(f;\Delta)$ accordingly.

Theorem 4. *Let the hypothesis be as in Theorem 1.*

(i) If $k > 0$ is even the Bruinier-Funke lift of F is a harmonic Maass form of weight $1/2 - k$ for $\Gamma_0(4p)$. The lift is a weakly holomorphic modular form if and only if the twisted L-function $L(\xi_{-2k}(F), -d, s)$ of $\xi_{-2k}(F) \in S_{3/2+k}(N)$ at $s = k+1$ vanishes. The Δ-th coefficient of its holomorphic part is given by

$$\frac{\sqrt{d}}{2\sqrt{\Delta}} \left(\frac{d}{4\pi\Delta}\right)^{k/2} \prod_{j=0}^{k/2-1} \left(\frac{k+1}{2} + j\right)\left(j - \frac{k}{2}\right) \left(\mathbf{t}^+_{-d}(F;\Delta) - \mathbf{t}^-_{-d}(F;\Delta)\right).$$

(ii) If k is odd the Bruinier-Funke lift of F is a weakly holomorphic modular form of weight $3/2 + k$ for $\Gamma_0(4p)$. The Δ-th coefficient of its holomorphic part is given by

$$\frac{\sqrt{d}}{2\sqrt{\Delta}} \left(-\frac{4\pi\Delta}{d}\right)^{(k+1)/2} \left(\mathbf{t}^+_{-d}(F;\Delta) - \mathbf{t}^-_{-d}(F;\Delta)\right).$$

(iii) If $k = 0$ the Bruinier-Funke lift of F is a harmonic Maass form of weight $1/2$ for $\Gamma_0(4p)$. The Δ-th coefficient of its holomorphic part is given by

$$\frac{\sqrt{d}}{2\sqrt{\Delta}} \left(\mathbf{t}^{+}_{-d}(F;\Delta) - \mathbf{t}^{-}_{-d}(F;\Delta)\right).$$

Remark 5. If $k = 0$ and the constant coefficients of the input function F do not vanish at all cusps, the Bruinier-Funke lift of F is a harmonic Maass form that maps to a linear combination of unary theta functions of weight $1/2$ under ξ_0. We show this by computing the lift of the non-holomorphic Eisenstein series of weight 0.

Example 6. We obtain

$$\mathcal{I}^{\mathrm{BF}}_{-3}(\tau, J) = f_3 = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 - 4096248q^9 + \cdots.$$

The two theta lifts satisfy a duality similar to Zagier's functions f_d and g_Δ. Let $\kappa = 3/2 + k$ if k is odd and $\kappa = 1/2 - k$ if k is even. We can realize the Fourier coefficients of harmonic Maass forms f of weight κ as traces of CM values of weight $-2k$ harmonic weak Maass forms F by showing that the Kudla-Millson lift is orthogonal to cusp forms and then using a pairing defined by Bruinier and Funke [BF04]. Analogous formulas hold for the Bruinier-Funke theta lift when κ is replaced by $\widetilde{\kappa}$ which is $3/2 + k$ if k is even and $1/2 - k$ if k is odd.

Remark 7. By considering $\mathcal{I}^{\mathrm{KM}}_{\Delta}(\tau, J)$ and $f = f_d$ or $\mathcal{I}^{\mathrm{BF}}_{-d}(\tau, J)$ and $f = g_\Delta$ as above we recover the relation of the coefficients of g_Δ and f_d. Note that our assumptions on Δ and d imply the equality of the two trace functions in this case.

We now describe how these results lead to nonvanishing conditions for the twisted central derivatives of L-functions of elliptic curves.

Elliptic curves and modular forms

Let E be an elliptic curve over $\mathbb{Q}$ given by the equation

$$E\colon\ y^2 = 4x^3 - 60G_4(\Lambda_E)x - 140G_6(\Lambda_E),$$

where $G_{2k}(\Lambda_E) := \sum_{w\in\Lambda_E\setminus\{0\}} w^{-2k}$ is the classical weight $2k$ Eisenstein series.

The elliptic curve E is isomorphic (over $\mathbb{C}$) to $\mathbb{C}/\Lambda_E$, where Λ_E is a lattice in $\mathbb{C}$. The corresponding isomorphism is called the analytic parametrization of E and is given by the map $t \mapsto P_t = (\wp(\Lambda_E;t), \wp'(\Lambda_E;t))$ for $t \in \mathbb{C}\setminus\Lambda_E$, where

$$\wp(\Lambda_E;t) := \frac{1}{t^2} + \sum_{w\in\Lambda_E\setminus\{0\}} \left(\frac{1}{(t-w)^2} - \frac{1}{w^2}\right)$$

is the usual Weierstrass $\wp$-function for Λ_E.

By a theorem of Mordell it is known that the group $E(\mathbb{Q})$ of rational points of E is a finitely generated abelian group, i.e. $E(\mathbb{Q}) = E(\mathbb{Q})^{\mathrm{tors}} \oplus \mathbb{Z}^r$ with $r \in \mathbb{Z}_{\geq 0}$ and $E(\mathbb{Q})^{\mathrm{tors}}$ finite. The Birch and Swinnerton-Dyer Conjecture relates the rank r to the analytic properties of the L-function $L(E, s)$ of E. More precisely, Birch and Swinnerton-Dyer conjectured that

$$L(E, s) = c \cdot (s-1)^r + \textit{higher order terms}$$

with $c \neq 0$ and $r = \mathrm{rank}(E)$.

The conjecture is true in the case that the analytic rank is equal to 0 or 1 by the work of Gross–Zagier, Kolyvagin and Wiles [GZ86, Kol88, Wil95]. Wiles et al. proved that for every elliptic curve $E/\mathbb{Q}$ of conductor N_E there is a weight 2 cusp form $G_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n \in S_2(N_E)$ that satisfies

$$L(G_E, s) = L(E, s).$$

Here, $L(G_E, s) = \sum_{n=1}^{\infty} a_E(n) n^{-s}$ is the L-function of G_E.

Thus, results on L-functions of weight 2 cusp forms, which are often easier to obtain, apply to the corresponding L-functions of elliptic curves. This was also used by Gross and Zagier in their work on the Birch and Swinnerton-Dyer Conjecture.

A different connection was established by Waldspurger [Wal81] and Kohnen–Zagier [KZ81] who proved that half-integer weight modular forms serve as "generating series" for the central values of quadratic twists of modular L-functions. They showed that there is a weight 3/2 cusp form whose coefficients are essentially the square roots of $L(G_E, D, 1)$, where $L(G_E, D, s) = \sum_{n=1}^{\infty} \chi_D(n) a_E(n) n^{-s}$ for a negative fundamental discriminant D and the associated Kronecker character $\chi_D = \left(\frac{D}{\cdot}\right)$. This twisted modular L-function corresponds to the D-quadratic twist of the elliptic curve $E : y^2 = x^3 + ax + b$ given by $E_D : Dy^2 = x^3 + ax + b$.

Elliptic curves and harmonic Maass forms

Let $G \in S_2(N)$ be a cusp form of weight 2 and D be a fundamental discriminant. Bruinier and Ono [BO10] recently observed that the vanishing of $L(G, D, 1)$ and $L'(G, D, 1)$ is related to the vanishing and the algebraicity of the Fourier coefficients of weight 1/2 harmonic Maass forms.

In their work, Bruinier and Ono consider weight 1/2 harmonic Maass forms f whose image under $\xi_{1/2}$ is equal to a real multiple of a weight 3/2 cusp form g that maps to G under the Shimura correspondence. That is, we have the following picture

$$\begin{tikzcd} & G \in S_2(N) \\ f \in H^+_{1/2}(4N) \arrow[r, "\xi_{1/2}"] & g \in S_{3/2}(4N). \arrow[u, "\text{Shimura}"] \end{tikzcd} \tag{0.3}$$

Employing deep work of Shimura and Waldspurger they proved that the Fourier coefficients of the non-holomorphic part of f as above give exact formulas for $L(G, D, 1)$. Using the theory of Borcherds products and the Gross-Zagier Theorem they show that at the same time the coefficients of the holomorphic part of f encode the vanishing of the central derivatives $L'(G, D, 1)$.

An interesting question is if there is a canonical preimage under ξ_0 of $G \in S_2(N)$ in the diagram (0.3) and a lifting map $\mathcal{I}$ such that the completed diagram is commutative, i.e.

$$\begin{tikzcd} F \in H_0^+(N) \arrow[r, "\xi_0"] \arrow[d, "\mathcal{I}"] & \xi_0(F) \in S_2(N) \\ \mathcal{I}(F) \in H_{1/2}^+(4N) \arrow[r, "\xi_{1/2}"] & \xi_{1/2}(\mathcal{I}(F)) \in S_{3/2}(4N). \arrow[u, "\text{Shimura}"'] \end{tikzcd} \tag{0.4}$$

We answer this question in the affirmative. We construct such canonical preimages F under ξ_0 of weight 2 cusp forms that correspond to an elliptic curve and show that $\xi_{1/2}$ of the Bruinier-Funke lift of F is equal to the Shintani lift of $\xi_0(F)$ (up to a constant). Moreover, we show that the coefficients of $\mathcal{I}_{-d}^{\mathrm{BF}}(\tau, F)$ encode the vanishing of $L(G_E, \Delta, 1)$ and $L'(G_E, \Delta, 1)$, where $\Delta > 1$ is a fundamental discriminant. In this special setting we obtain the corresponding results for the L-function of the elliptic curve E.

Weierstrass harmonic Maass forms

In the case of a weight 2 cusp form $G_E \in S_2(N_E)$ corresponding to an elliptic curve E of conductor N_E over $\mathbb{Q}$ there is a canonical preimage of G_E arising from the analytic parametrization of E. This was first observed by Guerzhoy [Gue13, Gue] and later worked out explicitly in [AGOR] by Griffin, Ono and Rolen. Let Λ_E be the lattice associated to E via the analytic parametrization. The canonical preimage of G_E arises from the Weierstrass ζ-function

$$\zeta(\Lambda_E; t) := \frac{1}{t} + \sum_{w \in \Lambda_E \setminus \{0\}} \left(\frac{1}{t-w} + \frac{1}{w} + \frac{t}{w^2} \right),$$

that is essentially the antiderivative of the Weierstrass $\wp$-function

$$\wp(\Lambda_E; t) = -\zeta'(\Lambda_E; t).$$

Furthermore, we make use of the modular parametrization. We let $\mathcal{E}_E(t)$ be the Eichler integral of a cusp form G_E defined as

$$\mathcal{E}_E(z) := -2\pi i \int_z^{i\infty} G_E(\tau) d\tau = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} \cdot q^n.$$

Moreover, we let $S(\Lambda_E) := \lim_{s \to 0^+} \sum_{w \in \Lambda_E \setminus \{0\}} \frac{1}{w^2 |w|^{2s}}$. Eisenstein observed that the function

$$\zeta^*(\Lambda_E; t) = \zeta(\Lambda_E; t) - S(\Lambda)t - \frac{\pi}{a(\Lambda_E)} \bar{t}$$

is lattice invariant, where $a(\Lambda_E)$ is the area of the fundamental parallelogram for Λ_E. This implies that

$$\mathcal{W}_E^*(z) := \zeta^*(\Lambda_E, \mathcal{E}_E(z))$$

is modular of weight 0. We have the following theorem.

Theorem 8. *There is a modular function $M_E(z)$ on $\Gamma_0(N_E)$ with algebraic Fourier coefficients for which $\mathcal{W}_E^*(z) - M_E(z)$ is a harmonic Maass form of weight 0 on $\Gamma_0(N_E)$. We call the function $\mathcal{W}_E(z) = \mathcal{W}_E^*(z) - M_E(z)$ a Weierstrass harmonic Maass form.*

Elliptic curves and Weierstrass harmonic Maass forms

For simplicity, we now let $N_E = p_E$ be a prime. In the body of this thesis we will also consider the general case. We let E be an elliptic curve of conductor p_E over $\mathbb{Q}$ and $G_E \in S_2(p_E)$ be the associated cusp form. Moreover, let $g_E \in S_{3/2}(4p_E)$ be a cusp form that maps to G_E under the Shimura correspondence. Recall that $-d$ is a negative fundamental discriminant that is congruent to a square modulo $4p_E$.

Using the Bruinier-Funke theta lift of the harmonic Maass form $\mathcal{W}_E$ associated to the cusp form $G_E \in S_2(N_E)$ we construct a weight 1/2 harmonic Maass form

$$f_E(\tau) = \mathcal{I}_{-d}^{\mathrm{BF}}(\tau, \mathcal{W}_E).$$

Here, we chose the function $M_E(z)$ such that the principal parts of $\mathcal{W}_E$ at all cusps other than ∞ and the constant coefficient at ∞ vanish. For ease of exposition, we also assume that $\mathcal{W}_E$ is invariant under the Fricke involution and that it is normalized such that $\xi_0(\mathcal{W}_E) = G_E/||G_E||^2$.

Then the sign of the functional equation of $L(G_E, s) = L(E, s)$ is $\epsilon(E) = -1$. Therefore, $L(E, 1) = 0$. For the Fourier expansion of f_E we write

$$f_E(z) = f_E^+(z) + f_E^-(z) = \sum_{n \gg -\infty} c_E^+(n) q^n + \sum_{n<0} c_E^-(n) \Gamma\left(\frac{1}{2}, 4\pi\, |n|\, y\right) q^n.$$

Let $\Delta > 1$ be a fundamental discriminant. By Theorem 4, we have that the Δ-th coefficient of f_E is given by

$$\frac{\sqrt{d}}{2\sqrt{\Delta}} \left(\mathbf{t}_{-d}^+(F;\Delta) - \mathbf{t}_{-d}^-(F;\Delta)\right) = \frac{\sqrt{d}}{2\sqrt{\Delta}} \sum_{z \in Z_{-d}(\Delta)} F(z),$$

for the twisted Heegner divisor

$$Z_{-d}(\Delta) = \sum_{Q \in \Gamma_0(p) \backslash \mathcal{Q}_{-d\Delta, p_E}^+} \frac{\chi_\Delta(Q)}{|\overline{\Gamma_0(p)}_Q|} \alpha_Q - \sum_{Q \in \Gamma_0(p) \backslash \mathcal{Q}_{-d\Delta, p_E}^-} \frac{\chi_\Delta(Q)}{|\overline{\Gamma_0(p)}_Q|} \alpha_Q.$$

We relate this modular trace to a certain differential of the third kind associated to $Z_{-d}(\Delta)$.

Using results by Scholl [Sch86] on the algebraicity of such differentials and the action of the Hecke algebra, we show the following theorem.

Theorem 9. *We assume the notation above. Then the following are equivalent:*

(i) $\frac{\sqrt{d}}{2\sqrt{\Delta}} \sum_{z \in Z_{-d}(\Delta)} F(z)$ *is rational.*

(ii) Some non-zero multiple of the projection of the image of $Z_{-d}(\Delta)$ *in the Jacobian to the* G_E*-isotypical component is the divisor of a rational function.*

Recall that $L(G_E, \Delta, s)$ denotes the twisted L-function of G_E and that for the elliptic curve $E : y^2 = x^3 + ax + b$ we have $L(G_E, \Delta, s) = L(E_\Delta, s)$, where $E_\Delta : \Delta y^2 = x^3 + ax + b$.
Combining Theorem 9 with the Gross–Zagier formula we obtain part (ii) of the following theorem.

Theorem 10. *With the same notation as above the following are true:*
(i) If $\Delta < 0$ *is a fundamental discriminant for which* $\left(\frac{\Delta}{p_E}\right) = 1$*, then*

$$L(E_\Delta, 1) = 0 \quad \textit{if and only if} \quad c_E^-(\Delta) = 0.$$

(ii) If $\Delta > 0$ *is a fundamental discriminant for which* $\left(\frac{\Delta}{p_E}\right) = 1$*, then*

$$L'(E_\Delta, 1) = 0 \quad \textit{if and only if} \quad c_E^+(\Delta) \quad \textit{is in} \quad \mathbb{Q}.$$

Part (i) follows from the fact that $\xi_{1/2}(f_E) \in \mathbb{R} g_E$ by diagram (0.4) and by work of Kohnen [Koh85] that relates the coefficients of g_E to the twisted L-function of G_E.

Remark 11. Theorem 10 above gives a more intrinsic version of Bruinier's and Ono's main theorem [BO10, Theorem 7.8] since we directly relate the cusp form G_E to the half-integer weight form f_E. Moreover, the proof for the relation between the algebraicity of $c_E^+(\Delta)$ and the vanishing of $L'(E_\Delta, 1) = 0$ is independent of Bruinier's and Ono's work. In particular, it does not rely on the construction of a Borcherds product and might therefore be easier to generalize to higher weights.

Remark 12. Note that we could also phrase part (ii) of Theorem 10 in terms of the coefficients of the Kudla-Millson lift. Our proof of this part only relies on the property that the coefficients are given as twisted traces and the fact that these predict the vanishing of the associated Heegner divisor. However, we could not prove part (i) since there we rely on the commutative diagram (0.4). The fact that the coefficients of the holomorphic parts of the two lifts encode the same arithmetic information is also reflected in their duality as explained before.

Remark 13. Part (ii) of Theorem 10 also gives conditions for the algebraicity of periods of differentials of the first and second kind associated to $\mathcal{W}_E$ and the Heegner divisor $Z_{-d}(\Delta)$. We explain this at the end of Chapter 6.

For elliptic curves E that satisfy the assumptions in Theorem 10 we then have by work of Kolyvagin [Kol88] and Gross and Zagier [GZ86] on the Birch and Swinnerton-Dyer Conjecture, that for fundamental discriminants Δ:

(i) If $\Delta < 0$, $\left(\frac{\Delta}{p_E}\right) = 1$, and $c_E^-(\Delta) \neq 0$, then the rank of $E_\Delta(\mathbb{Q})$ is 0.

(ii) If $\Delta > 0$, $\left(\frac{\Delta}{p_E}\right) = 1$, and $c_E^+(\Delta)$ is not rational, then the rank of $E_\Delta(\mathbb{Q})$ is 1.

We now present an example illustrating our results.

The elliptic curve of conductor 37

We consider the elliptic curve of conductor 37 given by the equation

$$E:\ y^2 = 4x^3 - 4x + 1.$$

The sign of the functional equation of the L-function of E is -1 and $E(\mathbb{Q})$ has rank 1 (see for example [LMF13]). The q-expansion of $G_E \in S_2(37)$ is given by

$$G_E(z) = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + \cdots \in S_2^{new}\left(\Gamma_0(37)\right)$$

and using `Sage` [S+14] we find that the q-expansion of the Weierstrass harmonic Maass form $\mathcal{W}_E(z)$ is given by

$$\mathcal{W}_E^+(z) = q^{-1} + 1 + 2.1132...q + 2.3867...q^2 + 4.2201...q^3 + 5.5566...q^4 + 8.3547...q^5 + O(q^6).$$

We write

$$f_E = \mathcal{I}_{-3}(\tau, \mathcal{W}_E(z)) = \sum_{n \gg -\infty} c_E^+(n) q^n + f^-(\tau).$$

The table below illustrates Theorem 10, and its implications for ranks of elliptic curves. It was computed by Strömberg [BS12].

Δ	$c_E^+(\Delta)$	$L'(E_\Delta, 1)$	$\mathrm{rank}(E_\Delta(\mathbb{Q}))$
1	$-0.2817617849\ldots$	$0.3059997738\ldots$	1
12	$-0.4885272382\ldots$	$4.2986147986\ldots$	1
21	$-0.1727392572\ldots$	$9.0023868003\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1489	9	0	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4393	66	0	3

Stephan Ehlen numerically confirmed that

$$c_E^+(\Delta) = \frac{1}{2\sqrt{\Delta}} \left(\mathbf{t}_{-3}^+(\mathcal{W}_E(z); \Delta) - \mathbf{t}_{-3}^-(\mathcal{W}_E(z); \Delta)\right)$$

for Δ as in the table using `Sage` [S$^+$14].

The structure of this thesis

In the first two chapters, we introduce the setting and the important objects of this thesis. In particular, we define the notion of vector valued harmonic Maass forms. The third and fourth chapter are the technical heart of the thesis: here we investigate the analytic and automorphic properties of the Kudla-Millson and the Bruinier-Funke theta lift and compute their Fourier expansion of the holomorphic part. In the fifth chapter, we investigate the relation of the lifts to their dual spaces. The sixth chapter is devoted to the connection between elliptic curves, their L-functions, the Weierstrass harmonic Maass forms and the coefficients of their Bruinier-Funke theta lifts. In the seventh chapter, we consider various applications of the results and derive the theorems presented in the introduction. In the eighth chapter, we present some future projects building upon the results in this thesis.

1. Basic notation

In the first part of this chapter we introduce quadratic forms, quadratic spaces and lattices which are the basic objects underlying the setting in this thesis. In the second part we describe the special situation we will work in. The exposition in this chapter is brief and the statements are presented without proofs, but we give references where a detailed exposition can be found.

1.1. Quadratic forms and lattices

We start with a short introduction to the theory of quadratic forms, quadratic spaces and lattices. The standard references for these topics are [Kit93, Kne02, Ser73]. A good overview is also given in Bruinier's part of [BvdGHZ08].

Let R be a ring with unity 1 and let M be a finitely generated R-module.

Definition 1.1.1. A *quadratic form* on M is a map $Q : M \to R$ such that:

(i) $Q(rx) = r^2 Q(x)$ for all $r \in R$ and $x \in M$.

(ii) $(x, y) := Q(x + y) - Q(x) - Q(y)$ is a bilinear form.

We say that (x, y) is the *bilinear form associated to* Q. We call the pair (M, Q) a *quadratic module over* R. If $R = K$ is a field, then (M, Q) is called a *quadratic space over* K.

Note that in the case that 2 is invertible in R, the second condition implies the first one in Definition 1.1.1.

Example 1.1.2. We will later consider *binary quadratic forms*, i.e. quadratic forms in two variables. For a binary quadratic form Q of the form

$$Q(x, y) = ax^2 + bxy + cy^2, \quad a,\ b,\ c \in \mathbb{R},$$

we write $Q = [a, b, c]$. If a, b, c are integers, we call $[a, b, c]$ an *integral binary quadratic form.*

In the following let (M, Q) be a quadratic module over R.

Definition 1.1.3. (i) Let x, $y \in M$. If $(x, y) = 0$, we say that x and y are *orthogonal* to each other.

(ii) Let $U \subset M$ be a subset of M. We define the *orthogonal complement of* U by

$$U^{\perp} := \{x \in M \,:\, (x,u) = 0 \text{ for all } u \in U\}\,.$$

(iii) A quadratic module M is called *non-degenerate* if $M^{\perp} = \{0\}$.

(iv) A non-zero vector $x \in M$ is called *isotropic* if $Q(x) = 0$, and *anisotropic* if $Q(x) \neq 0$.

Definition 1.1.4. Let $(\tilde{M}, \tilde{Q})$ be another quadratic module over R. An R-linear map $\sigma : M \to \tilde{M}$ is called an *isometry* if σ is injective and

$$\tilde{Q}(\sigma(x)) = Q(x) \text{ for all } x \in M.$$

If σ is also surjective, M and $\tilde{M}$ are called *isometric*.

Definition 1.1.5. The *orthogonal group* $\mathrm{O}(M)$ of M is defined as the group of all isometries from M onto itself

$$\mathrm{O}(M) := \{\sigma : M \to M \,:\, \sigma \text{ isometry}\}.$$

The *special orthogonal group* $SO(M)$ is the subgroup

$$\mathrm{SO}(M) := \{\sigma \in \mathrm{O}(M) \,:\, \det(\sigma) = 1\}.$$

Example 1.1.6. Let r, s be non-negative integers. By $\mathbb{R}^{r,s}$ we denote the quadratic space over $\mathbb{R}^{r+s}$ with the quadratic form

$$Q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2$$

for $x = (x_1, \ldots, x_{r+s})$. We denote the orthogonal group of $\mathbb{R}^{r,s}$ by $\mathrm{O}^{r,s}(\mathbb{R})$.

Definition 1.1.7. Suppose that M has a basis $\mathcal{B} = (b_1, \ldots, b_n)$. The *Gram matrix* of Q corresponding to $\mathcal{B}$ is the matrix $G = G_Q = (g_{ij})_{1 \le i,j \le n}$, where $g_{ij} := (b_i, b_j)$.

We define the *determinant of* M by

$$\det(Q) = \det(M) = \det((M,Q)) := \det(G_Q).$$

The determinant $\det(Q)$ (if non-zero) is well defined as an element of $R^*/(R^*)^2$, where R^* denotes the group of units of R.

Proposition 1.1.8. *Let (V, Q) be an n-dimensional non-degenerate quadratic space over $\mathbb{R}$. There exist non-negative integers r, s, with $n = r + s$, such that (V, Q) is isometric to $\mathbb{R}^{r,s}$. The pair (r, s) is called the signature of V.*

Definition 1.1.9. A *lattice* L is a finitely generated non-degenerate quadratic module over $\mathbb{Z}$. A lattice L is called *integral* if the bilinear form (λ, μ) takes values in $\mathbb{Z}$ for all λ, $\mu \in L$. It is called *even* if $Q(\lambda) \in \mathbb{Z}$ for all $\lambda \in L$ and *unimodular* if $|\mathrm{disc}(L)|$, the class of $|\det(G_L)|$ in $\mathbb{Z}^*/(\mathbb{Z}^*)^2$, is equal to 1. A lattice element $\lambda \in L \setminus \{0\}$ is called *primitive* if it satisfies $\mathbb{Q}\lambda \cap L = \mathbb{Z}\lambda$.

Definition 1.1.10. For a lattice L we define its *dual lattice* L' as

$$L' := \{\lambda \in L \otimes_{\mathbb{Z}} \mathbb{Q} \,:\, (\lambda, \mu) \in \mathbb{Z} \text{ for all } \mu \in L\}.$$

In the following let (L, Q) be a lattice. The quadratic form Q on L induces a well-defined map $Q : V(\mathbb{R}) := L \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ via the assignment

$$Q(\lambda \otimes r) := r^2 Q(\lambda),\ \lambda \in L,\ r \in \mathbb{R}.$$

Then, $(V(\mathbb{R}), Q)$ is a quadratic space that contains L as a discrete subgroup.

Definition 1.1.11. Let $V(\mathbb{R}) = L \otimes_{\mathbb{Z}} \mathbb{R}$ as above and let $\mathcal{B} = (b_1, \ldots, b_n)$ be a basis of $V(\mathbb{R})$ with $L = \sum_{i=1}^n \mathbb{Z} b_i$. Then n is called the *rank of* L. The *signature of* L is given by the signature of $V(\mathbb{R})$.

Remark 1.1.12. Let L^- denote the lattice L equipped with the quadratic form $-Q$. Then L^- has signature (s, r) if L has signature (r, s).

From now on let L be an even lattice.

Definition 1.1.13. The *level of* L is defined as

$$\min\{N \in \mathbb{Z}_{>0} \,:\, NQ(\lambda) \in \mathbb{Z} \text{ for all } \lambda \in L'\}.$$

Lemma 1.1.14. *Let G be the Gram matrix of L. Then we have*

$$|L'/L| = |\det(G)|.$$

Moreover, L'/L is a finite abelian group which is called the discriminant group of L.

Lemma 1.1.15. *The quadratic form induces a well-defined map*

$$\begin{aligned} Q : \quad & L'/L \to \mathbb{Q}/\mathbb{Z} \\ & \lambda + L \mapsto Q(\lambda + L) := Q(\lambda) \pmod 1. \end{aligned}$$

Such a tuple $(L'/L, Q)$ is called the discriminant form of L.

Definition 1.1.16. The *orthogonal group* $\mathrm{O}(L'/L)$ consists of all group homomorphisms $\sigma : L'/L \to L'/L$ satisfying $Q(\sigma(x)) = Q(x)$ for all $x \in L'/L$.

1.2. A rational quadratic space of signature $(1, 2)$

In this thesis we consider a 3-dimensional rational quadratic space of signature $(1, 2)$ that is isotropic over $\mathbb{Q}$. This is the same setting as in [BF06] and [BO13] (the general setting for higher dimensional spaces can be found in Bruinier's chapter of [BvdGHZ08]).

Let $N > 0$ be an integer. We let V be the rational quadratic space

$$V := \left\{ \lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & -\lambda_1 \end{pmatrix} \in \mathbb{Q}^{2\times 2} \,:\, \operatorname{tr}(\lambda) = 0 \right\}$$

with the quadratic form $Q(\lambda) = N \det(\lambda)$. The associated bilinear form is given by $(\lambda, \mu) = -N \operatorname{tr}(\lambda \cdot \mu)$for λ, $\mu \in V$. The quadratic space V has signature $(1, 2)$.

1.2.1. A special lattice

We consider the lattice

$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \,:\, a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice corresponding to the bilinear form as above is given by

$$L' := \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \,:\, a, b, c \in \mathbb{Z} \right\}.$$

We identify the discriminant group $L'/L =: \mathcal{D}$ with $\mathbb{Z}/2N\mathbb{Z}$, together with the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $x \mapsto -x^2/4N$. The level of L is $4N$.

For a fundamental discriminant $\Delta \in \mathbb{Z}$ we will later consider the rescaled lattice ΔL together with the quadratic form $Q_\Delta(\lambda) := \frac{Q(\lambda)}{|\Delta|}$. The corresponding bilinear form is given by $(\cdot,\cdot)_\Delta = \frac{1}{|\Delta|}(\cdot,\cdot)$. The dual lattice of ΔL corresponding to $(\cdot,\cdot)_\Delta$ is equal to L' as above, independent of Δ. We denote the discriminant group $L'/\Delta L$ by $\mathcal{D}(\Delta)$. Note that $\mathcal{D}(1) = \mathcal{D}$ and $|\mathcal{D}(\Delta)| = |\Delta|^3 \, |\mathcal{D}| = 2N \, |\Delta|^3$.

The lattice L is intimately related to binary quadratic forms and the congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ as we are going to explain now following [BO10]. Recall that $\mathrm{SL}_2(\mathbb{Z}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} \,:\, ad - bc = 1\}$ and that $\Gamma_0(N)$ is defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,:\, c \equiv 0 \pmod{N} \right\}.$$

In general, $\Gamma_0(N)$ acts on the space V via conjugation

$$g.\lambda = g\lambda g^{-1}, \text{ for } g \in \Gamma_0(N),\ \lambda \in V.$$

The natural homomorphism $\mathrm{SO}(L) \to \mathrm{O}(L'/L)$ is surjective (here, $\mathrm{SO}(L)$ is defined as in Definition 1.1.5). We denote its kernel by $\Gamma(L)$. Let $\mathrm{SO}^+(L)$ be the intersection of $\mathrm{SO}(L)$ and the connected component of the identity of $\mathrm{SO}(V(\mathbb{R}))$. The group $\Gamma_0(N)$ takes L to itself and acts trivially on the discriminant group $\mathcal{D}$. However, in general it does not act trivially on $\mathcal{D}(\Delta)$.

Proposition 1.2.1 (Proposition 2.2 in [BO10])**.** *The image of* $\Gamma_0(N)$ *in* $\mathrm{SO}(L)$ *is equal* *to* $\Gamma(L) \cap \mathrm{SO}^+(L)$. *The image in* $\mathrm{SO}(L)$ *of the extension of* $\Gamma_0(N)$ *by all Atkin-Lehner involutions is equal to* $\mathrm{SO}^+(L)$. *(The Atkin-Lehner involutions are defined in Section 1.2.3.)*

For $m \in \mathbb{Q}$ and $h \in L'/L$, we consider the set

$$L_{m,h} = \{\lambda \in L + h \,:\, Q(\lambda) = m\}\,. \tag{1.2.1}$$

We can identify lattice elements of $L_{m,h}$ with integral binary quadratic forms as follows. For $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L_{m,h}$ we consider the matrix

$$M(\lambda) := \begin{pmatrix} a & b/2 \\ b/2 & Nc \end{pmatrix} = \lambda \cdot \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix}.$$

Then $M(\lambda)$ defines an integral binary quadratic form $[a, b, Nc]$ of discriminant $D = b^2 - 4Nac = 4NQ(\lambda)$ satisfying $b \equiv h \pmod{2N}$. Conversely, every integral binary quadratic form $[a, b, Nc]$ of discriminant $D = b^2 - 4Nac = 4NQ(\lambda)$ that satisfies $b \equiv h \pmod{2N}$ defines an element $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L_{m,h}$. The group $\Gamma_0(N)$ acts on elements $M(\lambda)$ via $g.M(\lambda) = gM(\lambda)g^t$, for $g \in \Gamma_0(N)$ and $M(\lambda)$ as above. The actions of $\Gamma_0(N)$ on L and on quadratic forms are compatible.

Using this correspondence one can show that, by reduction theory, if $m \neq 0$, the group $\Gamma_0(N)$ acts on $L_{m,h}$ with finitely many orbits.

1.2.2. The associated symmetric space

Let $G = \mathrm{Spin}(V) \simeq \mathrm{SL}_2$ viewed as an algebraic group over $\mathbb{Q}$ and write $\overline{\Gamma}$ for its image in $\mathrm{SO}(V) \simeq \mathrm{PSL}_2$. We let D be the associated symmetric space.

The group $\mathrm{SL}_2(\mathbb{Q})$ acts on V by conjugation

$$g.\lambda := g\lambda g^{-1},\ \lambda \in V,\ g \in \mathrm{SL}_2(\mathbb{Q}).$$

The space D can be realized as the Grassmannian of lines in $V(\mathbb{R})$ on which the quadratic form Q is positive definite,

$$D \simeq \{z \subset V(\mathbb{R}) \,:\, \dim z = 1 \text{ and } Q|_z > 0\}.$$

We can identify the Grassmannian D with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,:\, \Im(z) > 0\}$ as follows. We write $z = x + iy$, with $x,\ y \in \mathbb{R}$, and obtain an isomorphism between D and $\mathbb{H}$ by

$$z \mapsto \mathbb{R}\lambda(z), \tag{1.2.2}$$

where we pick as a generator for the associated positive line

$$\lambda(z) := \frac{1}{\sqrt{N}y} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}. \tag{1.2.3}$$

The group $G(\mathbb{R})$ acts on $\mathbb{H}$ by linear fractional transformations. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{R})$ these are given by $z \mapsto \frac{az+b}{cz+d}$. The isomorphism in (1.2.2) above is then $G(\mathbb{R})$-equivariant. In particular, $Q(\lambda(z)) = 1$ and $g.\lambda(z) = \lambda(gz)$ for $g \in G(\mathbb{R})$.

1.2.3. Cusps

We let M be the modular curve $\Gamma_0(N) \setminus D$. Via the isomorphism between $\mathbb{H}$ and D as in (1.2.2) it can be identified with the usual modular curve $Y_0(N) = \Gamma_0(N) \setminus \mathbb{H}$. The curve $Y_0(N)$ is not compact, but can be completed to a compact Riemann surface $X_0(N)$ by adding finitely many points, the cusps, to $Y_0(N)$. The cusps are given as the $\Gamma_0(N)$-equivalence classes of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, i.e. $X_0(N) = Y_0(N) \cup (\Gamma_0(N) \setminus \mathbb{P}^1(\mathbb{Q}))$.

For square-free N we can describe the cusps of the modular curve $X_0(N)$ in an explicit and convenient way (see [Sch, BO13]). In order to do so we define the Atkin-Lehner involutions. A good reference for this topic is also [DS05].

Definition 1.2.2. Let Q be an exact divisor of N, i.e. $Q|N$ and $(Q, N/Q) = 1$. Then we define the *Atkin-Lehner involution* W_Q^N by any matrix

$$W_Q^N = \begin{pmatrix} Q\alpha & \beta \\ N\gamma & Q\delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z},$$

with determinant Q.

Moreover, we define the *Fricke involution* W_N by

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Remark 1.2.3. The matrices W_Q^N are uniquely determined up to left multiplication by elements of $\Gamma_0(N)$.

For exact divisors Q, Q' of N we define

$$Q * Q' = \frac{Q \cdot Q'}{(Q, Q')^2}. \tag{1.2.4}$$

In the case of square-free N the cusps are represented by $\frac{1}{Q}$, where Q runs through the divisors of N. Note that two cusps $(a : c)$ and $(a' : c')$ are equivalent under $\Gamma_0(N)$ if and only if $(c, N) = (c', N)$. In particular, a complete set of representatives for the cusps of $\Gamma_0(N)$ is given by $W_Q^N \infty$, where Q runs through the divisors of N, i.e. the Atkin-Lehner involutions act transitively in this case.

We now describe how we can identify the set of isotropic lines $\mathrm{Iso}(V)$ in $V(\mathbb{Q})$ with $\mathbb{P}^1(\mathbb{Q})$ following the exposition in [Fun02, BF06]. The identification is given by the map

$$\psi : \mathbb{P}^1(\mathbb{Q}) \to \mathrm{Iso}(V), \quad \psi((\alpha : \beta)) = \mathrm{span}\left(\begin{pmatrix} \alpha\beta & \alpha^2 \\ -\beta^2 & -\alpha\beta \end{pmatrix}\right).$$

The map ψ is a bijection and $\psi(g(\alpha:\beta)) = g.\psi((\alpha:\beta))$ for $g \in \Gamma_0(N)$. So the cusps of M can be identified with the $\Gamma_0(N)$-classes of $\mathrm{Iso}(V)$.

If we set $\ell_\infty := \psi(\infty)$, then ℓ_∞ is spanned by $\lambda_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. For $\ell \in \mathrm{Iso}(V)$ we pick $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma_\ell \ell_\infty = \ell$. Furthermore, we orient all lines ℓ by requiring that $\lambda_\ell := \sigma_\ell \lambda_\infty$ is a positively oriented basis vector of ℓ.

Let Γ_ℓ be the stabilizer of the line ℓ. Then

$$\sigma_\ell^{-1}\Gamma_\ell\sigma_\ell = \left\{ \pm \begin{pmatrix} 1 & k\alpha_\ell \\ 0 & 1 \end{pmatrix} \,:\, k \in \mathbb{Z} \right\},$$

where $\alpha_\ell \in \mathbb{Q}_{>0}$ is the width of the cusp ℓ [Fun02]. In our case it does not depend on the choice of σ_ℓ. For each ℓ there is a $\beta_\ell \in \mathbb{Q}_{>0}$ such that $\left(\begin{smallmatrix} 0 & \beta_\ell \\ 0 & 0 \end{smallmatrix}\right)$ is a primitive element of $\ell_\infty \cap \sigma_\ell L$. We write $\epsilon_\ell = \alpha_\ell/\beta_\ell$.

1.2.4. Heegner divisors

We define a divisor on the modular curve $M = \Gamma_0(N)\backslash D$ as follows. For a vector $\lambda \in V(\mathbb{Q})$ of positive norm, we let

$$D_\lambda = \mathrm{span}(\lambda) \in D.$$

For example, if $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L'/L$, the corresponding point in $\mathbb{H}$ is given by the solution of

$$\left(\begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix}, \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \right) = 0$$

that lies in the upper half plane, i.e. the point $z = \frac{b+\sqrt{b^2-4acN}}{2cN}$. (Here we used that $\mathbb{H}$ may be identified with the projective model of the symmetric space D, see for example [BvdGHZ08, Chapter 2.4].)

We denote the image of D_λ in M by $Z(\lambda)$. Then $Z(\lambda)$ is called *Heegner point*. We define a *Heegner divisor* $Z(m,h)$ by

$$Z(m,h) = \sum_{\lambda \in \Gamma_0(N)\backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|} Z(\lambda).$$

Here, $\overline{\Gamma}_\lambda$ denotes the stabilizer of λ in $\overline{\Gamma_0(N)}$, the image of $\Gamma_0(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$.

1.2.5. Geodesics

If $Q(\lambda) < 0$, we obtain a geodesic c_λ in D via

$$c_\lambda = \{ z \in D \,:\, z \perp \lambda \}.$$

We denote $\Gamma_\lambda \backslash c_\lambda$ in M by $c(\lambda)$.

There are two cases (see [Fun02]):

(i) We have $Q(\lambda) \notin -\frac{1}{4N}(\mathbb{Q}^*)^2$.

(ii) We have $Q(\lambda) \in -\frac{1}{4N}(\mathbb{Q}^*)^2$.

In the first case $Z(\lambda)$ is a closed geodesic in $Y_0(N)$ and in the second case it is an infinite geodesic in $Y_0(N)$ (see [Fun02] or [Höv12, Section 1.9] for more details).

Thus, if $c(\lambda)$ is an infinite geodesic, λ is orthogonal to two isotropic lines $\ell_\lambda = \mathrm{span}(\mu)$ and $\tilde{\ell}_\lambda = \mathrm{span}(\tilde{\mu})$, with μ and $\tilde{\mu}$ positively oriented. We fix an orientation of V and we say that ℓ_λ is the line associated to λ if the triple $(\lambda, \mu, \tilde{\mu})$ is a positively oriented basis for V. In this case, we write $\lambda \sim \ell_\lambda$.

These geodesics will be important when we describe the Fourier expansion of the Kudla-Millson theta lift in Chapter 3.

2. Automorphic Forms

In this chapter we recall the theory of scalar valued and vector valued modular forms. We introduce the notion harmonic weak Maass forms that was developed by Bruinier and Funke in [BF04] and we present various aspects of the theory of (vector valued) harmonic weak Maass forms. Moreover, we recall results of the author and Ehlen [AE13] on a method for twisting automorphic forms with a certain genus character related to the lattice L we defined in the previous chapter. We also define Poincaré series and Whittaker functions as well as the theta functions, that we will use as kernel functions for the lifts of harmonic weak forms in the upcoming chapters.

Throughout this thesis, we will use z and τ as variables in the upper half plane $\mathbb{H}$. We will use z when talking about integer weight forms, since these will later correspond to automorphic forms on the Grassmannian D. We will use τ as the symplectic variable corresponding to half-integer weight forms. We write q for $e^{2\pi i z}$ and for $e^{2\pi i\tau}$. It will be clear from the context whether $q = e^{2\pi i z}$ or $q = e^{2\pi i \tau}$.

2.1. Scalar valued modular forms

We briefly introduce the notion of scalar valued modular forms following the classical references for the theory [DS05, Kob93, KK07]. The books of Ono [Ono04] and Zagier [BvdGHZ08] also give a good overview and provide many interesting applications of the theory of modular forms.

Let N be a positive integer. Recall that the level N congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We define the *Petersson slash operator* for an integer $k \in \mathbb{Z}$ and a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(z) := (cz+d)^{-k} \det(\gamma)^{k/2} f(\gamma z). \tag{2.1.1}$$

Definition 2.1.1. Let $k \in \mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight k for* $\Gamma_0(N)$ if the following hold:

(i) $(f|_k\gamma)(z) = f(z)$ for all $\gamma \in \Gamma_0(N)$.

(ii) f is holomorphic at the cusps of $\Gamma_0(N)$.

If f vanishes at all cusps, it is called a *cusp form* and if it is only meromorphic at the cusps it is called a *weakly holomorphic modular form.*

By $M_k(N)$, $S_k(N)$, respectively $M_k^!(N)$ we denote the spaces of modular forms, cusp forms, respectively weakly holomorphic modular forms of weight k for $\Gamma_0(N)$.

Condition (i) implies that f is invariant under the transformation $z \mapsto z+1$, so f has a Fourier expansion of the form (by Condition (ii))

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

at the cusp ∞. Here, $q = e^{2\pi iz} = e(z)$. The function f has a Fourier expansion of a similar form at the other cusps of $X_0(N)$. If f is a cusp form, we require $a_f(0) = 0$. If f is weakly holomorphic, then finitely many coefficients of negative index occur, i.e. $a_f(n)$ might be 0 for finitely many $n < 0$.

2.1.1. Hecke operators and Atkin-Lehner involutions

We introduce Hecke operators, natural linear operators that act on spaces of modular forms.

Definition 2.1.2. Let p be a prime and $p \nmid N$. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(N)$, then we define the action of the *Hecke operator* $T(p)$ on $f(z)$ for p by

$$(f|_k T(p))(z) := \sum_{n=0}^{\infty} \left(a(pn) + p^{k-1} a(n/p)\right) q^n.$$

If $p \nmid n$, then $a(n/p) = 0$.

Remark 2.1.3. For the definition of $T(l)$ for any integer l see [DS05, Kob93].

Proposition 2.1.4. *Let $f \in M_k(N)$. For $p \geq 2$ we have $(f|_k T(p))(z) \in M_k(N)$. The Hecke operators take cusp forms to cusp forms.*

We recall some facts from the Atkin-Lehner theory of newforms.

Let $d > 1$. A cusp form $f \in S_k(N)$ is easily seen to be contained in $S_k(dN)$. A second way to embed $S_k(N)$ into $S_k(dN)$ is the so-called V–operator $V(d)$ defined on the Fourier expansion of a form $g(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ which lies in $M_k(N)$ by

$$\left(\sum_{n=n_0}^{\infty} a(n)q^n\right) |_k V(d) := \sum_{n=n_0}^{\infty} a(n) q^{dn}.$$

Then it is

$$F(dz) = (f|_k V(d))(z) \in S_k(\Gamma_0(dN)).$$

We now describe how to distinguish "old" forms in $S_k(dN)$ from forms that are "new" at this level, i.e. whose minimal level is dN.

We first define the subspace $S_k^{\mathrm{old}}(N)$ of forms whose minimal level is not dN by

$$S_k^{\mathrm{old}}(N) := \sum_{dM|N} S_k(M)|_k V(d),$$

where we sum over pairs of positive integers (d, M) for which $dM|N$ and $M \neq N$.

There is an inner product on the space of modular forms, called the *Petersson inner product*. For cusp forms $f, g \in S_k(N)$ the Petersson inner product is defined by

$$(f,g)_k := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \cdot \int_{\mathcal{F}_N} f(z)\overline{g(z)} y^k d\mu(z),$$

where $\mathcal{F}_N$ denotes a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$ and $d\mu(z) = \frac{dxdy}{y^2}$ denotes the invariant measure on $\mathbb{H}$.

Recall the definition of the Atkin-Lehner involution W_Q^N as in Definition 1.2.2. Then

$$W_Q^N = \begin{pmatrix} Q\alpha & \beta \\ N\gamma & Q\delta \end{pmatrix} \in \mathbb{Z}^{2\times 2},$$

where Q is an exact divisor of N and $\det(W_Q^N) = Q$.

If $f \in M_k(N)$, then $f \mapsto f|_k W_Q^N$ is independent of the choices α, β, γ, δ and defines an involution of $M_k(N)$.

Definition 2.1.5. The *subspace of newforms* $S_k^{\mathrm{new}}(N)$ is defined to be the orthogonal complement of $S_k^{\mathrm{old}}(N)$ in $S_k(N)$ with respect to the Petersson inner product.

A *newform* is a normalized eigenform in $S_k^{\mathrm{new}}(N)$ with respect to the Hecke operators and all of the Atkin-Lehner involutions $|_k W_Q^N$, where Q runs through the exact divisors of N, and $|_k W_N^N$.

Remark 2.1.6. Using the theory of Hecke operators one can show that every space of newforms has a basis of newforms. Furthermore, newforms determine distinct Hecke eigenspaces. This is known as the "multiplicity one" phenomenon.

2.1.2. L-functions of modular forms

For a cusp form $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{2k}(N)$ of weight $2k$ for $\Gamma_0(N)$ we define its L-function by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s},$$

where $s \in \mathbb{C}$.

These functions satisfy the following properties.

Proposition 2.1.7. *Let $f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{2k}(N)$. Then the following hold:*

(i) $L(f,s)$ is holomorphic for $s \in \mathbb{C}$ with $\Re(s) > k+1$.

(ii) $L(f,s)$ has an analytic continuation to all of $\mathbb{C}$.

(iii) Let $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$, where $\Gamma(s)$ denotes the usual Γ-function. Then

$$\Lambda(f, 2k-s) = (-1)^k w_N \Lambda(f,s)$$

for all $s \in \mathbb{C}$. Here, w_N is the eigenvalue of f under the Fricke involution, that is $f\left(-\frac{1}{Nz}\right) = w_N N^k z^{2k} f(z)$.

Let D be a fundamental discriminant and $\chi_D = \left(\frac{D}{\cdot}\right)$ be the associated Kronecker character . Then we define the twisted L-function of f by

$$L(f,D,s) = \sum_{n=1}^{\infty} \chi_D(n) \frac{a_f(n)}{n^s}.$$

Remark 2.1.8. The twisted L-function satisfies similar properties as $L(f,s)$. Especially, it has an analytic continuation to all of $\mathbb{C}$. Moreover, the completed twisted L-function $\Lambda(f,D,s) := (2\pi)^{-s}(ND^2)^{s/2}\Gamma(s)L(f,D,s)$ satisfies

$$\Lambda(f,D,s) = (-1)^k \left(\frac{D}{-N}\right) w_N \, \Lambda(f,D,2k-s),$$

where w_N is as before.

2.2. The Weil representation

Here, we give an overview of the Weil representation attached to an even lattice L. The exposition follows the one in [Bru02] and [Bor98].

We let $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ be the upper half plane. Recall that we will let $\tau \in \mathbb{H}$, and write $\tau = u + iv$ with u, $v \in \mathbb{R}$, throughout this thesis. We denote by $\sqrt{w} = w^{1/2}$ the principal branch of the square root, such that $\arg(w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Moreover, we put $w^{\alpha} := e^{\alpha \operatorname{Log}(w)}$ for $\alpha \in \mathbb{C}$, where $\operatorname{Log}(\cdot)$ denotes the principal branch of the logarithm. We set $e(w) := e^{2\pi i w}$.

Recall that the special linear group $\mathrm{SL}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2} : \det(A) = 1\}$ acts on $\mathbb{H}$ via fractional linear transformations

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \ \gamma \in \mathrm{SL}_2(\mathbb{R}).$$

We let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic group. It is the double cover of $\mathrm{SL}_2(\mathbb{R})$. For an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have two choices of the holomorphic square root of $\tau \mapsto c\tau + d$.

The elements of $\mathrm{Mp}_2(\mathbb{R})$ are therefore given by pairs (γ, ϕ), where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function satisfying $\phi(\tau)^2 = c\tau + d$. The multiplication of two elements in $\mathrm{Mp}_2(\mathbb{R})$ is given by

$$(\gamma, \phi(\tau))(\gamma', \phi'(\tau)) = (\gamma\gamma', \phi(\gamma'\tau)\phi'(\tau)).$$

The map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right)$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$.

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. We recall the following well-known lemma.

Lemma 2.2.1. *The group* $\mathrm{Mp}_2(\mathbb{Z})$ *is generated by*

$$T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \qquad \text{and} \qquad S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

The center of $\mathrm{Mp}_2(\mathbb{Z})$ *is given by*

$$Z := \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right).$$

We have $S^2 = (ST)^3 = Z$.

We let $\Gamma_\infty := \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \,:\, n \in \mathbb{Z} \}$ and set

$$\widetilde{\Gamma}_\infty := \langle T \rangle = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right) \,:\, n \in \mathbb{Z} \right\}.$$

From now on we let (L, Q) be an even lattice of signature (b^+, b^-) with dual lattice L'. Recall that L'/L is a finite abelian group.

Definition 2.2.2. The finite dimensional *group algebra* $\mathbb{C}[L'/L]$ *of* L is defined as the set of formal linear combinations $\sum_{h \in L'/L} a_h \mathfrak{e}_h$ with $a_h \in \mathbb{C}$. Here, the symbols $\mathfrak{e}_h$ are called *standard basis vectors* of the group algebra $\mathbb{C}[L'/L]$.

We define an inner product on $\mathbb{C}[L'/L]$ by

$$\left\langle \sum_{h \in L'/L} a_h \mathfrak{e}_h, \sum_{h \in L'/L} b_h \mathfrak{e}_h \right\rangle := \sum_{h \in L'/L} a_h \overline{b_h}.$$

Proposition 2.2.3. *There is a unitary representation* ρ_L *of* $\mathrm{Mp}_2(\mathbb{Z})$ *on the group algebra*

$\mathbb{C}[L'/L]$ which is defined by the action of the generators S, $T \in \mathrm{Mp}_2(\mathbb{Z})$ by

$$\begin{aligned}\rho_L(T)\mathfrak{e}_h &= e(Q(h))\mathfrak{e}_h\\ \rho_L(S)\mathfrak{e}_h &= \frac{\sqrt{i}^{(b^- - b^+)}}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h,h'))\mathfrak{e}_{h'}.\end{aligned}$$

The representation ρ_L is called the Weil representation attached to L.

Remark 2.2.4. We denote by ρ_L^* the dual representation of ρ_L and by $\overline{\rho}_L$ its complex conjugate. Then we have $\rho_L^* = \overline{\rho}_L = \rho_{L^-}$ with L^- as in Remark 1.1.12 [Shi75].

Using orthogonality relations between characters it is not hard to show the following lemma.

Lemma 2.2.5. *The standard generator Z of the center of $\mathrm{Mp}_2(\mathbb{Z})$ acts on $\mathfrak{e}_h$ as follows*

$$\rho_L(Z)\mathfrak{e}_h = e\left(\frac{b^- - b^+}{4}\right)\mathfrak{e}_{-h}.$$

Remark 2.2.6. The Weil representation ρ_L factors through the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ - b^-$ is even, where N is the level of L. If $b^+ - b^-$ is odd, it factors through a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Let h, $h' \in L'/L$ and $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$. We define the coefficient $\rho_{hh'}(\gamma,\phi)$ of the representation ρ_L by

$$\rho_{hh'}(\gamma,\phi) = \langle \rho_L(\gamma,\phi)\mathfrak{e}_{h'}, \mathfrak{e}_h\rangle.$$

The following proposition of Shintani [Shi75, Proposition 1.6] gives a formula for $\rho_{hh'}(\gamma,\phi)$. (Here, $\delta_{i,j}$ denotes the usual Kronecker delta.)

Proposition 2.2.7. *Let h, $h' \in L'/L$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then the coefficient $\rho_{hh'}(\gamma,\phi)$ is given by*

$$\sqrt{i}^{(b^- - b^+)(1-\operatorname{sgn}(d))}\delta_{h,ah'}(abQ(h)),$$

if $c = 0$, and by

$$\frac{\sqrt{i}^{(b^- - b^+)\operatorname{sgn}(c)}}{|c|^{\frac{b^- + b^+}{2}}\sqrt{|L'/L|}} \sum_{r \in L/cL} e\left(\frac{a(h+r,h+r) - 2(h',h+r) + d(h',h')}{2c}\right),$$

if $c \neq 0$.

2.3. Vector valued harmonic Maass forms

In this section we define vector valued automorphic forms. In particular, we introduce harmonic weak Maass forms, a new type of automorphic form introduced in the fundamental article [BF04] by Bruinier and Funke. In contrast to the forms studied by Maaß these are allowed to have linear exponential growth at the cusps.

Let (V, Q) be a rational quadratic space of signature (b^+, b^-) and let L be an even lattice.

We define the *Petersson slash operator* for $k \in \frac{1}{2}\mathbb{Z}$ and $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ on functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ by

$$\left(f|_{k,\rho_L}(\gamma, \phi)\right)(\tau) = \phi(\tau)^{-2k}\rho_L(\gamma, \phi)^{-1}f(\gamma\tau).$$

Definition 2.3.1. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form of weight k with respect to the representation ρ_L and the group* $\mathrm{Mp}_2(\mathbb{Z})$ if

(i) $(f|_{k,\rho_L}(\gamma, \phi))(\tau) = f(\tau)$ for all $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.

(ii) $\Delta_k f = 0$, where

$$\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

is the weight k hyperbolic Laplace operator.

(iii) There is a $C > 0$ such that $f(\tau) = O(e^{Cv})$ as $v \to \infty$ uniformly in u, where $\tau = u+iv$.

We denote the space of these forms by H_{k,ρ_L} and define a subspace $H^+_{k,\rho_L} \subset H_{k,\rho_L}$ by replacing condition (iii) by

(iii') There is a Fourier polynomial

$$P_f(\tau) := \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Z}+Q(h)\\ -\infty\ll n\leq 0}} c^+_f(n, h)q^n\mathfrak{e}_h,$$

such that

$$f(\tau) - P_f(\tau) = O(e^{-\epsilon v}) \quad \text{as } v \to \infty,$$

for some constant $\epsilon > 0$. Here, $q = e^{2\pi i\tau}$. Then $P_f(\tau)$ is called the *principal part* of f.

From now on we will frequently omit the word "weak" from the definition. It will also be clear from the context if a harmonic Maass form is in H^+_{k,ρ_L} or H_{k,ρ_L}.

We write f_h for the h-th component of a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$, i.e.

$$f = \sum_{h\in L'/L} f_h\mathfrak{e}_h.$$

Remark 2.3.2. We denote the spaces of weakly holomorphic modular forms, modular forms, and cusp forms by $M^!_{k,\rho_L}$, M_{k,ρ_L}, respectively S_{k,ρ_L}. Holomorphic functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ are annihilated by the weight k Laplace operator, so we have the inclusions

$$H_{k,\rho_L} \supset H^+_{k,\rho_L} \supset M^!_{k,\rho_L} \supset M_{k,\rho_L} \supset S_{k,\rho_L}.$$

By A_{k,ρ_L} we denote the space of functions that transform of weight k with respect to the representation ρ_L (without requiring any analytic properties). We call this the space of *automorphic forms* (note that some authors require not only the correct transformation behavior but also some analytic conditions in the definition of automorphic forms).

Remark 2.3.3. For a unimodular, even lattice we recover the definition of scalar valued modular forms (for $\mathrm{SL}_2(\mathbb{Z})$) as in Section 2.1. By adjusting the analytic properties appropriately in Definition 2.1.1 we can also define scalar valued harmonic Maass forms for the group $\Gamma_0(N)$. We denote the space of such forms by $H_k(N)$, respectively $H^+_k(N)$.

Remark 2.3.4. Let N be the level of L. Using Remark 2.2.6 it is not hard to show that the components of $f \in M_{k,\rho_L}$ are scalar valued modular forms. If $b^+ + b^-$ is even, then the components f_h are modular forms for $\Gamma(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N},\, b \equiv c \equiv 0 \pmod{N}\}$. If $b^+ + b^-$ is odd, then $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and the components f_h transform with respect to slightly more complicated multiplier systems (see for example [Kob93]).

As in the scalar valued case, a function $f \in H_{k,\rho_L}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{h \in L'/L} \sum_{n \in \mathbb{Q}} c_f(n, h, v) q^n.$$

Since $\Delta_k f = 0$, the coefficients $c(n, h, v)$ satisfy the differential equation $\Delta_k c(n, h, v) = 0$ as functions in v. Computing the space of solutions to this differential equation gives rise to the following proposition.

Proposition 2.3.5. *Let $f \in H_{k,\rho_L}$ with $k \neq 1$. Then f uniquely decomposes as $f = f^+ + f^-$, with*

$$f^+(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n \gg -\infty}} c^+_f(n, h) q^n \mathfrak{e}_h \tag{2.3.1}$$

and

$$f^-(\tau) = \sum_{h \in L'/L} \left(c^-_f(0, h) v^{1-k} + \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n \neq 0}} c^-_f(n, h) W(2\pi n v) q^n \right) \mathfrak{e}_h, \tag{2.3.2}$$

where

$$W(x) := W_k(x) = \int_{-2x}^{\infty} e^{-t}t^{-k}dt.$$

If $x < 0$, then $W(x) = \Gamma(1-k, 2|x|)$, where $\Gamma(a,x)$ is the incomplete Γ-function. The function f^+ is called the holomorphic and f^- the non-holomorphic part of f.

If $f \in H^+_{k,\rho_L}$, the Fourier expansion of f^- is given by

$$f^-(\tau) = \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} c_f^-(n,h)W(2\pi nv)q^n\mathfrak{e}_h. \tag{2.3.3}$$

Remark 2.3.6. In the setting of this thesis as described in Section 1.2, we obtain a $\mathbb{C}[\mathbb{Z}/2N\mathbb{Z}]$-valued function with Fourier expansion (assuming that $f \in H^+_{k,\rho_L}$ and $k < 1$)

$$f^+(\tau) = \sum_{h\,(2N)} \sum_{\substack{n\in\mathbb{Z}\\ n\gg-\infty}} c_f^+(n,h)q^{\frac{n}{4N}}\mathfrak{e}_h,$$

and

$$f^-(\tau) = \sum_{h\,(2N)} \sum_{\substack{n\in\mathbb{Z}\\ n<0}} c_f^-(n,h)\Gamma\left(1-k, 4\pi\left|\frac{n}{4N}\right|v\right)q^{\frac{n}{4N}}\mathfrak{e}_h.$$

The operation of Z as in Lemma 2.2.5 implies that the Fourier coefficients of $f \in H_{k,\rho_L}$ satisfy

$$c_f^{\pm}(n,h) = (-1)^{k+\frac{b^- - b^+}{2}}c_f^{\pm}(n,-h).$$

From this we directly deduce the following corollary.

Corollary 2.3.7. *The space H_{k,ρ_L} is trivial if*

$$2k \not\equiv b^+ - b^- \pmod 2.$$

2.3.1. Differential operators acting on automorphic forms

We introduce some differential operators acting on the space of harmonic Maass forms. The basic reference here is [Bum98]. We define the differential operators $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial\bar{\tau}}$ by

$$\frac{\partial}{\partial\tau} := \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \qquad \text{and} \qquad \frac{\partial}{\partial\bar{\tau}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Definition 2.3.8. Let $k \in \frac{1}{2}\mathbb{Z}$. We define the Maass raising and lowering operators on smooth functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ by

$$R_k := 2i\frac{\partial}{\partial\tau} + kv^{-1} \qquad \text{and} \qquad L_k := -2iv^2\frac{\partial}{\partial\bar{\tau}}.$$

The lowering operator L_k takes automorphic forms of weight k to automorphic forms of weight $k-2$ and the raising operator R_k takes automorphic forms of weight k to automorphic forms of weight $k+2$. Moreover, these operators commute with the slash operator [Bum98, Lemma 2.1.1].

We can write Δ_k in terms of L_k and R_k as follows

$$-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k.$$

The Maass raising and lowering operators satisfy the following relations with the weighted Laplace operator

$$R_k\Delta_k = (\Delta_{k+2} - k)R_k, \tag{2.3.4}$$

$$L_k\Delta_k = (\Delta_{k-2} + 2 - k)L_k. \tag{2.3.5}$$

We also define iterated versions of the raising and lowering operators

$$R_k^n := R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k, \qquad L_k^n := L_{k-2(n-1)} \circ \cdots L_{k-2} \circ L_k.$$

For $n = 0$ we set $R_k^0 = L_k^0 = \mathrm{id}$.

Using (2.3.4) and (2.3.5) we can show that the iterated versions of the Maass raising and lowering operators satisfy relations similar to the ones that R_k and L_k satisfy.

Lemma 2.3.9. *For $k \in \mathbb{Z}$ we have*

$$\Delta_0 R^k_{-2k} = R^k_{-2k}\left(\Delta_{-2k} - k(k+1)\right).$$

If k is even, then

$$\Delta_{3/2+k}R_{3/2}^{k/2} = R_{3/2}^{k/2}\left(\Delta_{3/2} + \frac{k}{4}(k+1)\right),$$

$$\Delta_{1/2-k}L_{1/2}^{k/2} = L_{1/2}^{k/2}\left(\Delta_{1/2} + \frac{k}{4}(k+1)\right),$$

and if k is odd we have

$$\Delta_{1/2-k}L_{3/2}^{(k+1)/2} = L_{3/2}^{(k+1)/2}\left(\Delta_{3/2} + \frac{k}{4}(k+1)\right),$$

$$\Delta_{3/2+k}R_{1/2}^{(k+1)/2} = R_{1/2}^{(k+1)/2}\left(\Delta_{1/2} + \frac{k}{4}(k+1)\right).$$

Following Bruinier and Funke [BF04] we consider another important differential operator whose most important features will be discussed in the next section.

Definition 2.3.10. Let $f \in H_{k,\rho_L}$ be a harmonic Maass form. We define the antilinear differential operator ξ_k by

$$\xi_k(f)(\tau) := v^{k-2}\overline{L_k f(\tau)} = R_{-k} v^k \overline{f(\tau)} = 2iv^k \overline{\frac{\partial}{\partial \bar{\tau}} f(\tau)}$$

and the usual Dolbeault operators ∂ and $\bar{\partial}$ on 1-forms, by requiring

$$\begin{aligned}
\partial(f d\tau + g d\bar{\tau}) &= \left(\frac{\partial}{\partial \tau} g\right) d\tau \wedge d\bar{\tau} \\
\bar{\partial}(f d\tau + g d\bar{\tau}) &= \left(\frac{\partial}{\partial \bar{\tau}} f\right) d\bar{\tau} \wedge d\tau.
\end{aligned}$$

We denote by $\mathcal{E}^k$ the space of C^∞-differential k-forms. Then $d = \partial + \bar{\partial}$ for the exterior derivative $d : \mathcal{E}^1 \to \mathcal{E}^2$.

We summarize some useful identities between the various differential operators in the following lemma.

Lemma 2.3.11. *We have*

$$\bar{\partial}(f d\tau) = -v^{2-k}\overline{\xi_k(f)} d\mu(\tau) = -L_k f d\mu(\tau).$$

We introduce another differential operator

$$D := \frac{1}{2\pi i} \frac{\partial}{\partial \tau}.$$

Then we have the following lemma.

Lemma 2.3.12 (Lemma 2.1 in [BOR08])**.** *We have*

$$D^{k-1} = \frac{1}{(-4\pi)^{k-1}} R_{2-k}^{k-1}.$$

Moreover, Bruinier, Ono and Rhoades proved the following theorem.

Theorem 2.3.13 (Theorem 1.1 in [BOR08])**.** *Let* $2 \le k \in \mathbb{Z}$ *and* $f \in H^+_{-2k}(N)$ *with Fourier expansion as in* (2.3.3)*. Then we have*

$$D^{k-1}(f) \in M^!_k(N)$$

and

$$D^{k-1} f = D^{k-1} f^+ = \sum_{n \gg -\infty} c_f^+(n) n^{k-1} q^n.$$

Moreover, the constant terms of $D^{k-1}(f)$ *at all cusps of* $\Gamma_0(N)$ *vanish.*

A short calculation yields the following lemma.

Lemma 2.3.14. *For a C^∞-function F on $X_0(N)$ we have*

$$dF = -\frac{1}{2i}\overline{\xi_0(F)}d\bar{\tau} + 2\pi i D(F)d\tau. \tag{2.3.6}$$

2.3.2. Harmonic Maass forms and the ξ-operator

In [BF04] Bruinier and Funke show that the ξ-operator relates harmonic Maass forms to modular forms.

Proposition 2.3.15 (Proposition 3.2 and Theorem 3.7 in [BF04])**.** *The operator ξ_k defines a surjective mapping*

$$\xi_k : H_{k,\rho_L} \to M^!_{2-k,\overline{\rho}_L}.$$

The kernel of this map is given by $M^!_{k,\rho_L}$.

Remark 2.3.16. The space H^+_{k,ρ_L} can alternatively be defined as

$$H^+_{k,\rho_L} := \left\{ f \in H_{k,\rho_L} \,:\, \xi_k(f) \in S_{2-k,\overline{\rho}_L} \right\}.$$

A direct consequence of Proposition 2.3.15 is the following corollary.

Corollary 2.3.17. *The following sequences are exact*

$$0 \longrightarrow M^!_{k,\rho_L} \longrightarrow H_{k,\rho_L} \xrightarrow{\xi_k} M^!_{2-k,\overline{\rho}_L} \longrightarrow 0, \tag{2.3.7}$$

$$0 \longrightarrow M^!_{k,\rho_L} \longrightarrow H^+_{k,\rho_L} \xrightarrow{\xi_k} S_{2-k,\overline{\rho}_L} \longrightarrow 0. \tag{2.3.8}$$

Via a direct computation we obtain the following lemma.

Lemma 2.3.18. *Let $k \neq 1$. For $f \in H_{k,\rho_L}$ the Fourier expansion of $\xi_k(f) \in M^!_{2-k,\overline{\rho}_L}$ is given by*

$$\xi_k(f) = -\sum_{h \in L'/L} \left(\overline{c_f^-(0,h)}(k-1) + \sum_{n \in \mathbb{Z}+Q(h)} \overline{c_f^-(-n,h)}(4\pi n)^{1-k} q^n \right) \mathfrak{e}_h.$$

Let f and g be automorphic forms of weight k transforming with respect to the representation ρ_L. We define the regularized Petersson inner product of f and g as

$$(f,g)^{\mathrm{reg}}_{k,\rho_L} = \lim_{t\to\infty} \int_{\mathcal{F}_t} \langle f(\tau), g(\tau) \rangle v^k d\mu(\tau),$$

whenever this expression exists. Here, $\mathcal{F}_t = \{\tau \in \mathcal{F} \,:\, \Im(\tau) \leq t\}$ denotes the truncated fundamental domain.

In [BF04] Bruinier and Funke define a bilinear pairing between the spaces $M_{2-k,\overline{\rho}_L}$ and H^+_{k,ρ_L} by

$$\{g, f\} = (g, \xi_k(f))^{\mathrm{reg}}_{2-k,\overline{\rho}_L}, \tag{2.3.9}$$

where $g \in M_{2-k,\overline{\rho}_L}$ and $f \in H^+_{k,\rho_L}$. They obtain the following duality result.

Proposition 2.3.19 (Proposition 3.5, [BF04])**.** *Let $g \in M_{2-k,\overline{\rho}_L}$ with Fourier expansion $g = \sum_{h,n} b_g(n,h)q^n\mathfrak{e}_h$, and $f \in H^+_{k,\rho_L}$ with Fourier expansion as in (2.3.1) and (2.3.3). Then the above defined pairing of g and f is determined by the principal part of f. It is equal to*

$$\{g, f\} = \sum_{h\in L'/L}\sum_{n\geq 0} c_f^+(n,h)b_g(-n,h).$$

We extend the pairing between $M_{2-k,\overline{\rho}_L}$ and H^+_{k,ρ_L} to include not only holomorphic modular forms but also weakly holomorphic modular forms.

Proposition 2.3.20. *For $g(\tau) = \sum_{h\in L'/L}\sum_{j\gg-\infty}^{\infty} b_g(j,h)q^j\mathfrak{e}_h \in M^!_{2-k,\overline{\rho}_L}$ and $f \in H^+_{k,\rho_L}$ with Fourier expansion as in* (2.3.1) *and* (2.3.3) *we have*

$$\{g, f\} = \sum_{h\in L'/L}\left(\sum_{n\geq 0} c_f^+(n,h)b_g(-n,h) + \sum_{n>0} c_f^+(-n,h)b_g(n,h)\right).$$

Proof. We follow the argument of Bruinier and Funke [BF04, Proposition 3.5].

First, we note that $\langle g, \bar{f}\rangle d\tau$ is a $\mathrm{Mp}_2(\mathbb{Z})$-invariant 1-form on $\mathbb{H}$. By Lemma 2.3.11 we have

$$d\left(\langle g, \bar{f}\rangle d\tau\right) = \bar{\partial}\left(\langle g, \bar{f}\rangle d\tau\right) = -\langle g, \overline{L_k f}\rangle d\mu(\tau)$$

and obtain using Stoke's theorem

$$\int_{\mathcal{F}_t} \langle g, \overline{L_k f}\rangle d\mu(\tau) = -\int_{\partial\mathcal{F}_t} \langle g, \bar{f}\rangle d\tau.$$

Since the integrand is $\mathrm{SL}_2(\mathbb{Z})$-invariant the equivalent pieces of the boundary of the fundamental domain cancel and we obtain

$$-\int_{\partial\mathcal{F}_t} \langle g, \bar{f}\rangle d\tau = \int_{-1/2}^{1/2} \langle g(u+it), \overline{f(u+it)}\rangle du.$$

If we insert the Fourier expansions of g and f, we observe that the integral picks out the 0-th Fourier coefficient of $\langle g, \overline{f}\rangle$. Therefore

$$\int_{\mathcal{F}_t} \langle g, \overline{L_k f}\rangle d\mu = \sum_{h\in L'/L}\sum_{n\in\mathbb{Z}+Q(h)} c_f^+(n,h)b_g(-n,h) + O(e^{-\epsilon t})$$

for some $\epsilon > 0$. Taking the limit as $t \to \infty$ we obtain

$$\{g, f\} = \lim_{t\to\infty} \int_{\mathcal{F}_t} \langle g, \overline{L_k f}\rangle d\mu = \sum_{h\in L'/L} \sum_{n\in\mathbb{Z}+Q(h)} c_f^+(n,h) b_g(-n,h).$$

□

Proposition 2.3.21 (Proposition 3.11 in [BF04]). *For every Fourier polynomial of the form*

$$P(\tau) = \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n<0}} c^+(n,h) q^n \mathfrak{e}_h$$

with $c^+(n,h) = (-1)^{k+\frac{b^- - b^+}{2}} c^+(n,-h)$, *there is an* $f \in H^+_{k,\rho_L}$ *with principal part* $P_f(\tau) = P(\tau) + \mathfrak{c}$ *for some* T*-invariant constant* $\mathfrak{c} \in \mathbb{C}[L'/L]$. *If* $k < 0$, *then* f *is uniquely determined.*

From the non-degeneracy of the bilinear pairing (2.3.9) and the formula in Proposition 2.3.19 we deduce the following corollary.

Corollary 2.3.22. *A harmonic Maass form* $f \in H^+_{k,\rho_L}$ *with constant principal part must satisfy* $f^- \equiv 0$. *In this case* f *is a modular form in* M_{k,ρ_L}.

This implies that $f, \widetilde{f} \in H^+_{k,\rho_L}$ with the same principal part actually have the same non-holomorphic part as well, i.e. $f^- = \widetilde{f}^-$. Therefore, the non-holomorphic part f^- of a harmonic Maass form $f \in H^+_{k,\rho_L}$ is determined by the principal part P_f of f. Obviously, the converse is wrong, since ξ_k is surjective.

2.3.3. Jacobi forms and Hecke operators

We briefly introduce Jacobi forms. These forms were first investigated by Eichler and Zagier in [EZ85]. There is a natural isomorphism from the space of Jacobi forms to the space of vector valued modular forms. Using this isomorphism we are able to define Hecke operators on vector valued modular forms.

Definition 2.3.23. Let $k, m \in \mathbb{Z}$. A holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is called a *holomorphic Jacobi form of weight* k *and index* m if

(i) $\phi\left(\gamma\tau, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e(mcz^2/(c\tau+d))\phi(\tau,z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(ii) $\phi(\tau, z + r\tau + s) = e(-m(r^2\tau + 2rz))\phi(\tau,z)$ for all $r, s \in \mathbb{Z}$.

(iii) $\phi(\tau,z)$ is holomorphic at the cusp ∞.

We denote the space of holomorphic Jacobi forms of weight k and index m by $J_{k,m}$. Such a function $\phi \in J_{k,m}$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n \zeta^r,$$

where $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$. If $\phi(\tau, z)$ is holomorphic at ∞, then $c(n,r) = 0$ if the discriminant $r^2 - 4nm$ is positive.

Proposition 2.3.24. *Let $\phi \in J_{k,m}$. Then $\phi(\tau, z)$ has a theta expansion of the form*

$$\phi(\tau, z) = \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \phi_r(\tau)\theta_r(\tau, z),$$

where

$$\theta_r(\tau, z) = \sum_{\substack{n\in\mathbb{Z}\\ n\equiv r(2m)}} q^{\frac{n^2}{4m}} \zeta^n$$

for $r \in \mathbb{Z}/2m\mathbb{Z}$.

If $L = \mathbb{Z}$ is the lattice with the quadratic form $Q(x) = mx^2$, then

$$\Psi(\tau) = \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \phi_r(\tau)\mathfrak{e}_r$$

is a vector valued modular form contained in $M_{k-1/2,\rho_L}$. This correspondence gives an isomorphism

$$M_{k-1/2,\rho_L} \simeq J_{k,m}.$$

Remark 2.3.25. If L is the lattice defined in Section 1.2.1 and $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then we have $M_{k,\rho_L} \simeq J_{k+1/2,N}$.

Remark 2.3.26. The space $M_{k,\overline{\rho}_L}$ is isomorphic to the space of skew-holomorphic Jacobi forms of weight $k + 1/2$ and index N as defined in [Sko90a].

Remark 2.3.27. The space $S^{\mathrm{new}}_{k,\rho_L}$ is isomorphic as a module over the Hecke algebra to $S^{\mathrm{new},-}_{2k-1}(N)$. This is the space of newforms on which the Fricke involution acts by multiplication with $(-1)^{k+1/2}$. The isomorphism is given by the Shimura correspondence. Analogously, we can relate $S^{\mathrm{new}}_{k,\overline{\rho}_L}$ to $S^{\mathrm{new},+}_{2k-1}(N)$. Here, the Fricke involution acts by multiplication with $(-1)^{k-1/2}$.

Using these isomorphisms we can introduce the operation of the Hecke algebra that was introduced for the space of Jacobi forms in [EZ85]. For any positive integer ℓ we have a Hecke operator $T(\ell)$ on M_{k,ρ_L} that is self adjoint with respect to the Petersson inner product (note that here ρ_L can be either ρ_L or $\overline{\rho}_L$). The action on the Fourier expansion of a vector valued modular form $g(\tau) = \sum_{h,n} b(n,h) q^n \mathfrak{e}_h \in M_{k,\rho_L}$ can be described explicitly. Moreover, this action extends to harmonic weak Maass forms.

If p is a prime that is coprime to N, we have $g|_k T(p) = \sum_{h,n} b^*(n,h) q^n \mathfrak{e}_h$, where

$$b^*(n,h) = b(p^2 n, ph) + p^{k-3/2} \left(\frac{4N\sigma n}{p} \right) b(n,h) + p^{2k-2} b(n/p^2, h/p), \tag{2.3.10}$$

and where $\sigma = 1$ if $\Delta < 0$ and $\sigma = -1$ if $\Delta > 0$.

Bruinier and Ono proved a series of useful results regarding the action of Hecke operators on the bilinear pairing which we recall now.

Proposition 2.3.28 (Proposition 7.1, [BO10])**.** *The action of the Hecke operator is self adjoint with respect to the pairing* $\{\cdot,\cdot\}$ *(up to a scalar factor). In particular, we have*

$$\{g, f|_k T(l)\} = l^{2k-2} \{g|_{2-k} T(l), f\}$$

for $g \in S_{2-k,\rho_L}$ *and* $f \in H^+_{k,\bar{\rho}_L}$.

Lemma 2.3.29 (Lemma 7.2, [BO10])**.** *Let* $g \in S_{2-k,\rho_L}$ *and assume that* $f \in H^+_{k,\bar{\rho}_L}$ *satisfies* $\{g, f\} = 1$, *and* $\{g', f\} = 0$ *for all* $g' \in S_{2-k,\rho_L}$ *orthogonal to* g. *Then* $\xi_k(f) = ||g||^{-2} g$.

Let F be a number field. Denote by $S_{2-k,\rho_L}(F)$ the F-vector space of cusp forms having coefficients in F. Moreover, we write $H^+_{k,\bar{\rho}_L}(F)$ for the space of harmonic weak Maass forms whose principal part has coefficients in F.

Lemma 2.3.30 (Lemma 7.3, [BO10])**.** *Let* $g \in S_{2-k,\rho_L}(F)$. *Then there is an* $f \in H^+_{k,\bar{\rho}_L}(F)$ *such that*

$$\xi_k(f) = ||g||^{-2} g.$$

Lemma 2.3.31 (Lemma, 7.4, [BO10])**.** *Let* $f \in H^+_{k,\rho_L}$ *and assume that* $\xi_k(f)|_{2-k} T(l) = \lambda_l \xi_k(f)$ *with* $\lambda_l \in F$. *Then*

$$f|_k T(l) - l^{2k-2} \lambda_l f \in M^!_{k,\rho_L}(F).$$

2.4. Twisting automorphic forms

In this section we recall work of Ehlen and the author [AE13] on twisting automorphic forms with a genus character related to the lattice L defined in Section 1.2. We let the setting be as in Section 1.2. Since the lattice L is fixed from now on, we write ρ for ρ_L (here and in the following).

We first define a generalized genus character for $\delta = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L'$ as in [GKZ87]. From now on we let $\Delta \in \mathbb{Z}$ be a fundamental discriminant and r an integer such that $\Delta \equiv r^2 \pmod{4N}$.

Then

$$\chi_\Delta(\delta) = \chi_\Delta([a,b,Nc]) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta | b^2 - 4Nac,\ (b^2-4Nac)/\Delta \text{ is a} \\ & \text{square mod } 4N \text{ and } \gcd(a,b,c,\Delta) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $[a, b, Nc]$ is the integral binary quadratic form corresponding to δ, and n is any integer prime to Δ represented by one of the quadratic forms $[N_1 a, b, N_2 c]$ with $N_1 N_2 = N$ and $N_1, N_2 > 0$.

The function χ_Δ is invariant under the action of $\Gamma_0(N)$ and under the action of all Atkin-Lehner involutions. It can be computed via the following formula [GKZ87, Section I.2, Proposition 1]: If $\Delta = \Delta_1\Delta_2$ is a factorization of Δ into discriminants and $N = N_1 N_2$ is a factorization of N into positive factors such that $(\Delta_1, N_1 a) = (\Delta_2, N_2 c) = 1$, then

$$\chi_\Delta([a, b, Nc]) = \left(\frac{\Delta_1}{N_1 a}\right)\left(\frac{\Delta_2}{N_2 c}\right).$$

If no such factorizations of Δ and N exist, we have $\chi_\Delta([a, b, Nc]) = 0$.

Since $\chi_\Delta(\delta)$ depends only on $\delta \in L'$ modulo ΔL, we can view it as a function on the discriminant group $\mathcal{D}(\Delta)$. Let ρ_Δ be the representation corresponding to the discriminant group $\mathcal{D}(\Delta)$.

In [AE13] it was shown that we obtain an intertwiner of the Weil representations corresponding to $\mathcal{D} = L'/L$ and $\mathcal{D}(\Delta)$ via χ_Δ. This idea was due to Stephan Ehlen.

Proposition 2.4.1 (Proposition 3.2,[AE13])**.** *We denote by $\pi : \mathcal{D}(\Delta) \to \mathcal{D}$ the canonical projection. For $h \in \mathcal{D}$, we define*

$$\psi_{\Delta,r}(\mathfrak{e}_h) := \sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh\\ Q_\Delta(\delta)\equiv \mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} \chi_\Delta(\delta)\mathfrak{e}_\delta. \tag{2.4.1}$$

Then $\psi_{\Delta,r} : \mathcal{D} \to \mathcal{D}(\Delta)$ defines an intertwining linear map between the representations $\widetilde{\rho}$ and ρ_Δ, where

$$\widetilde{\rho} = \begin{cases} \rho & \text{if } \Delta > 0, \\ \overline{\rho} & \text{if } \Delta < 0. \end{cases}$$

Remark 2.4.2. For a function $f \in A_{k,\rho_\Delta}$ Proposition 2.4.1 directly implies that the function $g : \mathbb{H} \to \mathbb{C}[\mathcal{D}]$, $g = \sum_{h\in\mathcal{D}} g_h\mathfrak{e}_h$ with $g_h := \langle\psi_{\Delta,r}(\mathfrak{e}_h), f\rangle$, is contained in $A_{k,\widetilde{\rho}}$.

2.5. Poincaré series and Whittaker functions

We recall some facts on Poincaré series with exponential growth at the cusps following Section 2.6 of [BO13]. Again we work in the setting that we introduced in Section 1.2.

We let $k \in \frac{1}{2}\mathbb{Z}$, and $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ denote the usual Whittaker functions (see p. 190 of [AS84]). For $s \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$ we put

$$\mathcal{M}_{s,k}(y) = y^{-k/2} M_{-\frac{k}{2}, s-\frac{1}{2}}(y).$$

We let Γ_∞ be the subgroup of $\Gamma_0(N)$ generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. For an integer $k \in \mathbb{Z}$, $m \in \mathbb{Z}_{>0}$,

$z = x + iy \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, we define

$$F_m(z, s, k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} [\mathcal{M}_{s,k}(4\pi m y) e(-mx)] \,|_k\, \gamma. \tag{2.5.1}$$

This Poincaré series converges for $\Re(s) > 1$, and it is an eigenfunction of Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$. Its specialization at $s_0 = 1 - k/2$ is a harmonic Maass form [Bru02, Proposition 1.10]. The principal part at the cusp ∞ is given by $q^{-m} + C$ for some constant $C \in \mathbb{C}$. The principal parts at the other cusps are constant.

The Poincaré series behave nicely under the Maass raising and lowering operator.

Proposition 2.5.1 (Proposition 2.2, [BO13])**.** *We have that*

$$R_k F_m(z, s, k) = 4\pi m \left(s + \frac{k}{2} \right) F_m(z, s, k+2).$$

Proposition 2.5.2. *We have that*

$$L_k F_m(z, s, k) = \frac{1}{4\pi m} \left(s - \frac{k}{2} \right) F_m(z, s, k-2).$$

Proof. Since L_k commutes with the slash operator, it suffices to show the identity on the corresponding Whittaker functions. We employ equations (13.4.11) and (13.1.32) in [AS84] which imply the desired identity. □

We now define $\mathbb{C}[L'/L]$-valued analogs of these series. Let $h \in L'/L$ and $m \in \mathbb{Z} - Q(h)$ be positive. For $k \in \left(\mathbb{Z} - \frac{1}{2}\right)_{<1}$ we let

$$\mathcal{F}_{m,h}(\tau, s, k) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} [\mathcal{M}_{s,k}(4\pi m y) e(-mx) \mathfrak{e}_h]\big|_{k,\rho}\, \gamma,$$

and for $k \in \left(\mathbb{Z} - \frac{1}{2}\right)_{\geq 1}$ we let

$$\mathcal{F}_{m,h}(\tau, s, k) = \frac{1}{2} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} [\mathcal{M}_{s,k}(4\pi m y) e(-mx) \mathfrak{e}_h]\big|_{k,\rho}\, \gamma.$$

The series $\mathcal{F}_{m,h}(\tau, s, k)$ converges for $\Re(s) > 1$ and it defines a harmonic Maass form of weight k for the group $\mathrm{Mp}_2(\mathbb{Z})$ with representation ρ. The special value at $s = 1 - k/2$ if $k \in \left(\mathbb{Z} - \frac{1}{2}\right)_{<1}$, respectively $s = k/2$ if $k \in \left(\mathbb{Z} - \frac{1}{2}\right)_{\geq 1}$, is harmonic [Bru02, Proposition 1.10]. For $k \in \mathbb{Z} - \frac{1}{2}$ the principal part is given by $q^{-m}\mathfrak{e}_h + q^{-m}\mathfrak{e}_{-h} + \mathfrak{c}$ for some constant $\mathfrak{c} \in \mathbb{C}[L'/L]$.

Remark 2.5.3. For $k < 0$ these Poincaré series span the space $H^+_{k,\rho}$ [Bru02, Proposition 1.12]. For $k = 0$ we have to add the constants to obtain a basis for $H^+_{0,\rho}$.

We will also consider the $\mathcal{W}$-Whittaker function

$$\mathcal{W}_{s,k}(y) = y^{-k/2} W_{k/2,s-1/2}(y), \quad y > 0. \tag{2.5.2}$$

It behaves as follows under the Maass raising and lowering operator.

Proposition 2.5.4. *For $m > 0$ and $y > 0$ we have that*

$$L_k \mathcal{W}_{s,k}(4\pi my)e(-mx) = \frac{1}{4\pi m}\left(s - \frac{k}{2}\right)\left(1 - s - \frac{k}{2}\right)\mathcal{W}_{s,k-2}(4\pi my)e(-mx)$$

and

$$R_k \mathcal{W}_{s,k}(4\pi my)e(-mx) = (-4\pi m)\mathcal{W}_{s,k+2}(4\pi my)e(-mx).$$

Proof. For the first equation we use (13.1.33) and (13.4.23) and for the second one (13.1.33) and (13.4.26) in [AS84]. □

2.6. Theta Functions

In this section we introduce the theta functions that we will employ as kernel functions for the lifts we investigate in the upcoming chapters. We start with the definition of the usual Siegel theta function in a slightly more general setting than the one explained in Section 1.2. Apart from the Siegel theta function, we will define the relevant theta kernels only in the setting of this thesis as presented in Section 1.2. More general constructions can be found in [Bor98, KM86] or [BF04].

2.6.1. The Siegel theta function

Let (V, Q) be a rational quadratic space of signature (b^+, b^-) and let $L \subseteq V$ be an even lattice of full rank with dual lattice L'. We let $\mathrm{Gr}(L)$ be the space of b^+-dimensional positive definite subspaces of $L \otimes \mathbb{R}$, i.e.

$$\mathrm{Gr}(L) := \{z \subset V(\mathbb{R}) \, : \, \dim z = b^+, \, Q|_z > 0\}.$$

For $z \in \mathrm{Gr}(L)$ we let $z^\perp$ be its orthogonal complement in $V(\mathbb{R})$. Then $z^\perp$ is a b^--dimensional negative definite subspace of $V(\mathbb{R})$ and $V(\mathbb{R}) = z \oplus z^\perp$. Therefore, we can uniquely decompose any $\lambda \in V(\mathbb{R})$ as $\lambda = \lambda_z + \lambda_{z^\perp}$, where λ_z, respectively $\lambda_{z^\perp}$ denotes the orthogonal projection of λ to z respectively $z^\perp$. We let

$$Q_z(\lambda) := Q(\lambda_z) - Q(\lambda_{z^\perp})$$

be the majorant associated to $z \in \mathrm{Gr}(L)$. Then $Q_z(\lambda)$ is a positive definite quadratic form on $V(\mathbb{R})$ for all $z \in \mathrm{Gr}(L)$. Using this quadratic form we can employ the classical construction of theta functions for positive definite quadratic forms as in [FB93, KK07, BvdGHZ08] (this construction does not work for an indefinite lattice L since the resulting series does not converge).

Definition 2.6.1. The *Siegel theta function* $\vartheta_L(\tau, z)$ associated with the lattice L is defined as

$$\vartheta_L(\tau, z) := v^{\frac{b^-}{2}} \sum_{\lambda \in L} e\left(\frac{\lambda_z^2}{2}\tau + \frac{\lambda_{z^\perp}^2}{2}\bar{\tau}\right) = v^{\frac{b^-}{2}} \sum_{\lambda \in L} e\left(Q(\lambda)u + Q_z(\lambda)iv\right),$$

where $\tau = u + iv \in \mathbb{H}$ and $z \in \mathrm{Gr}(L)$.

Now let V be the rational quadratic space of signature $(1, 2)$ as in Section 1.2. Then the majorant is given by

$$Q_z(\lambda) = \frac{1}{2}\left((\lambda, \lambda(z))^2 - (\lambda, \lambda)\right),$$

where $\lambda(z)$ is as in (1.2.3). Note that for $\lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \in L'/L$ we have

$$(\lambda, \lambda(z)) = -\frac{1}{\sqrt{N}y}(cN|z|^2 - bx + a) \tag{2.6.1}$$

and

$$(\lambda, \lambda) = -\frac{b^2}{2N} + 2ac. \tag{2.6.2}$$

Furthermore, we define

$$R(\lambda, z) := \frac{1}{2}(\lambda, \lambda(z))^2 - (\lambda, \lambda).$$

We then let

$$\varphi_{\mathrm{S}}^0(\lambda, z) := e^{-2\pi R(\lambda, z)} \tag{2.6.3}$$

and

$$\varphi_{\mathrm{S}}(\lambda, \tau, z) := \varphi_{\mathrm{S}}^0(\sqrt{v}\lambda, z)e^{2\pi i Q(\lambda)\tau}. \tag{2.6.4}$$

Then the Siegel theta function for the lattice L is given by

$$\Theta_L(\tau, z, \varphi_{\mathrm{S}}) = v \sum_{h \in L'/L} \sum_{\lambda \in L+h} \varphi_{\mathrm{S}}(\lambda, \tau, z)\mathfrak{e}_h =: v \sum_{h \in L'/L} \sum_{\lambda \in L+h} \theta_h(\tau, z, \varphi_{\mathrm{S}})\mathfrak{e}_h.$$

It satisfies the following transformation properties (see for example [Bor98, Bru02]).

Theorem 2.6.2. *The Siegel theta function $\Theta_L(\tau, z, \varphi_{\mathrm{S}})$ is a non-holomorphic automorphic form of weight $-1/2$ transforming with respect to the representation ρ for $\mathrm{Mp}_2(\mathbb{Z})$ in the variable τ and a $\Gamma_0(N)$-invariant function in the variable z.*

We now explain the construction of the twisted theta function $\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{S}})$ using the methods of Section 2.4. Let ρ_Δ be the representation corresponding to the discriminant group $\mathcal{D}(\Delta)$. We define by

$$\varphi_{\mathrm{S},\Delta}^0(\lambda, z) := e^{-2\pi R(\lambda, z)/|\Delta|}$$

and
$$\varphi_{\mathrm{S},\Delta}(\lambda,\tau,z) := \varphi^0_{\mathrm{S},\Delta}(\sqrt{v}\lambda, z)e^{2\pi i Q_\Delta(\lambda)\tau}$$
the twisted versions of (2.6.3) and (2.6.4). We let $\delta \in \mathcal{D}(\Delta)$ and define a theta function $\Theta_\delta(\tau, z, \varphi_\mathrm{S})$ for $\tau, z \in \mathbb{H}$ via

$$\Theta_\delta(\tau, z, \varphi_\mathrm{S}) = \sum_{\lambda\in\Delta L+\delta} \varphi_{\mathrm{S},\Delta}(\lambda,\tau,z).$$

By Theorem 2.6.2 the vector valued theta series

$$\Theta_{\mathcal{D}(\Delta)}(\tau, z, \varphi_\mathrm{S}) = \sum_{\delta\in\mathcal{D}(\Delta)} \Theta_\delta(\tau, z, \varphi_\mathrm{S})\mathfrak{e}_\delta$$

is a non-holomorphic automorphic form of weight $-1/2$ which transforms with respect to the representation ρ_Δ in the variable τ.

We obtain a $\mathbb{C}[L'/L]$-valued twisted theta function by setting

$$\Theta_{\Delta,r}(\tau, z, \varphi_\mathrm{S}) := \sum_{h\in L'/L} \sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh\\ Q_\Delta(\delta)\equiv \mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} \chi_\Delta(\delta)\Theta_\delta(\tau, z, \varphi_\mathrm{S})\mathfrak{e}_h.$$

Recall that
$$\widetilde{\rho} = \begin{cases} \rho & \text{if } \Delta > 0,\\ \overline{\rho} & \text{if } \Delta < 0.\end{cases}$$

Proposition 2.4.1 then directly implies the following theorem.

Theorem 2.6.3. *The theta function $\Theta_{\Delta,r}(\tau, z, \varphi_\mathrm{S})$ is a non-holomorphic $\mathbb{C}[\mathcal{D}]$-valued modular form of weight $-1/2$ for the representation $\widetilde{\rho}$ in the variable τ. Furthermore, it is $\Gamma_0(N)$-invariant in the variable $z \in D$.*

2.6.2. The Millson theta function

We define a Schwartz function

$$\psi_{\mathrm{KM}}(\lambda,\tau,z) = p_z(\lambda)e^{-2\pi Q(\lambda_z)},$$

where $p_z(\lambda) = (\lambda, \lambda(z))$.

This function was recently studied extensively by Hövel in his PhD thesis [Höv12]. It can be understood in the context of Borcherds' definition of Siegel theta functions as in [Bor98]. Kudla and Millson were the first that considered this function [KM90]. We call it *Millson Schwartz function* throughout this thesis.

We let
$$\psi^0_{\mathrm{KM},\Delta}(\lambda, z) = p_z(\lambda)e^{-2\pi R(\lambda,z)/|\Delta|}, \tag{2.6.5}$$

and

$$\psi_{\mathrm{KM},\Delta}(\lambda,\tau,z) = \psi^0_{\mathrm{KM},\Delta}(\sqrt{v}\lambda, z)e^{2\pi i Q_\Delta(\lambda)\tau}.$$

The associated theta function, that we call Millson theta function, has the following automorphic and analytic properties.

Theorem 2.6.4. *The theta function*

$$\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}}) := v^{1/2}\sum_{h\in\mathcal{D}} \sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh\\ Q_\Delta(\delta)\equiv\mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} \chi_\Delta(\delta) \sum_{\lambda\in\Delta L+\delta} \psi_{\mathrm{KM},\Delta}(\lambda,\tau,z)\mathfrak{e}_h \tag{2.6.6}$$

is a non-holomorphic $\mathbb{C}[\mathcal{D}]$-valued modular form of weight $1/2$ for the representation $\widetilde{\rho}$ in the variable τ. Furthermore, it is $\Gamma_0(N)$-invariant in the variable $z \in D$.

Proof. This is Satz 2.6 and Satz 2.8 in [Höv12]. Alternatively it can be deduced from the automorphic properties of the theta functions Borcherds defined and investigated in [Bor98] using the methods of Section 2.4. □

Moreover, we have

$$\psi_{\mathrm{KM},\Delta}(g.\lambda, gz) = \psi_{\mathrm{KM},\Delta}(\lambda, z)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$.

The components of the Millson theta function are denoted by $\Theta_{\Delta,r,h}(\tau,z,\psi_{\mathrm{KM}})$. We investigate the growth of the Millson theta function at the cusps of $\Gamma_0(N)$.

Proposition 2.6.5. *For $h \in L'/L$ and for each cusp ℓ, we have*

$$\Theta_{\Delta,r,h}(\tau,\sigma_\ell z,\psi_{\mathrm{KM}}) = O(e^{-Cy^2}), \quad \text{as } y\to\infty,$$

uniformly in x, for some constant $C > 0$.

Proof. We proceed as Funke in [Fun02] when investigating the growth properties of the Kudla-Millson theta function φ_{KM}. For simplicity we let $\Delta = N = 1$ (note that the theta function vanishes in that case, nevertheless the argument we apply here works as in the more general cases). Then $L = \mathbb{Z}^3$ and $h = \left(\begin{smallmatrix} h' & 0 \\ 0 & h' \end{smallmatrix}\right)$ with $h' = 0$ or $h' = 1/2$. So we consider

$$\theta_h(\tau,z,\psi_{\mathrm{KM}}) = \sum_{\substack{a,c\in\mathbb{Z}\\ b\in\mathbb{Z}+h'}} -\frac{v}{y}(c|z|^2 - bx + a)e^{-\frac{\pi v}{y}(c|z|^2-bx+a)^2}e^{2\pi i\bar{\tau}(-b^2/4+ac)}.$$

We apply Poisson summation on the sum over a. We consider the summands as a function of a and compute the Fourier transform, i.e.

$$\begin{aligned} &-\int_{-\infty}^{\infty}\frac{v}{y}(c|z|^2-bx+a)e^{-\frac{\pi v}{y}(c|z|^2-bx+a)^2}e^{2\pi i\bar{\tau}(-b^2/4+ac)}e^{2\pi i w a}da \\ &= -ye^{-\pi i\bar{\tau}b^2/2}e^{2\pi i(c\bar{\tau}+w)(bx-c|z|^2)}\int_{-\infty}^{\infty} te^{-\pi t^2}e^{2\pi i t\frac{y}{\sqrt{v}}(c\bar{\tau}+w)}dt, \end{aligned}$$

where we set $t = \frac{\sqrt{v}}{y}(c|z|^2 - bx + a)$. Since the Fourier transform of $xe^{-\pi x^2}$ is $ixe^{-\pi x^2}$ this equals

$$\begin{aligned}
&-i\frac{y^2}{\sqrt{v}}e^{-\pi i\bar{\tau}b^2/2}e^{2\pi i(c\bar{\tau}+w)(bx-c|z|^2)}(c\bar{\tau}+w)e^{-\frac{\pi y^2}{v}(c\bar{\tau}+w)^2}\\
&= -i\frac{y^2}{\sqrt{v}}(c\bar{\tau}+w)e^{-2\pi i\bar{\tau}(b/2-cx)^2}e^{2\pi i(bxw-cx^2w)}e^{-\frac{\pi y^2}{v}|c\tau+w|^2}.
\end{aligned}$$

We obtain that

$$\theta_h(\tau,z,\psi_{\mathrm{KM}}) = -\frac{y^2}{\sqrt{v}}\sum_{\substack{w,c\in\mathbb{Z}\\ b\in\mathbb{Z}+h'}}(c\bar{\tau}+w)e^{-2\pi i\bar{\tau}(b/2-cx)^2}e^{2\pi i(bxw-cx^2w)}e^{-\frac{\pi y^2}{v}|c\tau+w|^2}.$$

If c and w are non-zero this decays exponentially, and if $c = w = 0$ it vanishes.

In general we obtain for $h \in L'/L$ and at each cusp ℓ

$$\theta_h(\tau,\sigma_\ell z,\psi_{\mathrm{KM}}) = O(e^{-Cy^2}), \quad \text{as } y \to \infty,$$

uniformly in x, for some constant $C > 0$. □

Hövel investigated the behavior of $\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}})$ under the operation of the Atkin-Lehner involutions.

Proposition 2.6.6 (Proposition 2.7 in [Höv12])**.** *Let Q be an exact divisor of N. Then we have*

$$\Theta_{\Delta,r}(\tau,W_Q^N z,\psi_{\mathrm{KM}}) = \sum_{h\in L'/L}\Theta_{\Delta,r,W_Q^N.h}(\tau,z,\psi_{\mathrm{KM}})\mathfrak{e}_h.$$

By ℓ and ℓ' we denote the primitive isotropic vectors

$$\ell = \begin{pmatrix}0 & 1/N\\ 0 & 0\end{pmatrix}, \qquad \ell' = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}$$

in L. We write K for the 1-dimensional lattice $\mathbb{Z}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \subset L$. We have $L = K + \mathbb{Z}\ell + \mathbb{Z}\ell'$ and $L'/L \simeq K'/K$. Then we can rewrite the Millson theta function in terms of the smaller lattice K. The following is Satz 2.22 in [Höv12].

Proposition 2.6.7. *Let $\epsilon = 1$ if $\Delta > 0$ and $\epsilon = i$ if $\Delta < 0$. We have*

$$\begin{aligned}
\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}}) = &-\frac{Ny^2\bar{\epsilon}}{2i}\sum_{n=1}^{\infty}n\left(\frac{\Delta}{n}\right)\\
&\times\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}\left[\frac{1}{v^{1/2}}e\left(-\frac{Nn^2y^2}{2i\,|\Delta|\,v}\right)\sum_{\lambda\in K'}e\left(\frac{\lambda^2}{2}\,|\Delta|\,\bar{\tau}-2nN\lambda x\right)\mathfrak{e}_{r\lambda}\right]\Bigg|_{1/2,\widetilde{\rho_K}}\gamma.
\end{aligned}$$

2.6.3. The Kudla-Millson theta function

Following Kudla and Millson [KM86] we define a Schwartz function valued in the differential forms of Hodge type $(1,1)$. We let

$$\varphi_{\mathrm{KM}}(\lambda, z) := \left((\lambda, \lambda(z))^2 - \frac{1}{2\pi}\right) e^{-2\pi Q(\lambda)_z}\Omega,$$

where $\Omega = \frac{i}{2}\frac{dz\wedge d\bar{z}}{y^2}$.

Mimicking the construction in the previous sections, we let

$$\varphi^0_{\mathrm{KM},\Delta}(\lambda, z) := \left(\frac{1}{|\Delta|}(\lambda, \lambda(z))^2 - \frac{1}{2\pi}\right) e^{-2\pi R(\lambda,z)/|\Delta|}\Omega,$$

and

$$\varphi_{\mathrm{KM},\Delta}(\lambda, \tau, z) = e^{2\pi i Q_\Delta(\lambda)\tau}\varphi^0_{\mathrm{KM},\Delta}(\sqrt{v}\lambda, z).$$

Using the transformation properties of the untwisted Kudla-Millson theta function (see [KM86, BF04, BF06]) and the method presented in Section 2.4 we obtain the following theorem.

Theorem 2.6.8 (Proposition 4.1 in [AE13])**.** *The Kudla-Millson theta function*

$$\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}}) = \sum_{h\in\mathcal{D}} \sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh\\ Q_\Delta(\delta)\equiv\mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} \chi_\Delta(\delta) \sum_{\lambda\in\Delta L+\delta} \varphi_{\mathrm{KM},\Delta}(\lambda, \tau, z)\mathfrak{e}_h$$

is a non-holomorphic $\mathbb{C}[\mathcal{D}]$-valued modular form of weight $3/2$ for the representation $\widetilde{\rho}$ in the variable τ. Furthermore, it is $\Gamma_0(N)$-invariant in the variable $z \in D$.

Moreover, we have

$$\varphi_{\mathrm{KM},\Delta}(g.\lambda, gz) = \varphi_{\mathrm{KM},\Delta}(\lambda, z) \tag{2.6.7}$$

for $g \in \mathrm{SL}_2(\mathbb{R})$ (see [KM86, BF06]).

Funke [Fun02] already investigated the growth of the Kudla-Millson theta function.

Proposition 2.6.9 (Proposition 4.1 in [Fun02], Proposition 4.1 in [BF06])**.** *For $h \in L'/L$ and for each cusp ℓ, we have*

$$\Theta_{\Delta,r,h}(\tau, \sigma_\ell z, \varphi_{\mathrm{KM}}) = O(e^{-Cy^2}), \quad \text{as } y \to \infty,$$

uniformly in x, for some constant $C > 0$.

Using the same arguments as Hövel in his thesis for the Millson theta function (see Proposition 2.6.6) we can show the following proposition regarding the operation of the Atkin-Lehner involutions on $\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}})$.

Proposition 2.6.10. *Let Q be an exact divisor of N. Then we have*

$$\Theta_{\Delta,r}(\tau, W_Q^N z, \varphi_{\mathrm{KM}}) = \sum_{h \in L'/L} \theta_{\Delta,r,W_Q^N.h}(\tau, z, \varphi_{\mathrm{KM}})\mathfrak{e}_h.$$

Again, we rewrite the theta function with respect to the smaller lattice K. We can either use the usual method of Poisson summation or twist the corresponding untwisted expression which is derived in [BF06]. An unpublished preprint of Ehlen [Ehl] explains how to twist the reduction of $\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}})$ to the smaller lattice K.

Proposition 2.6.11. *We have*

$$\begin{aligned}\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}}) = -y\frac{N^{3/2}}{2\,|\Delta|}\,\bar{\epsilon}\sum_{n=1}^{\infty}\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})} n^2\left(\frac{\Delta}{n}\right)\\ \times\left[\exp\left(-\pi\frac{y^2Nn^2}{v\,|\Delta|}\right)v^{-3/2}\sum_{\lambda\in K'} e\left(|\Delta|\,Q(\lambda)\bar{\tau} - 2N\lambda n x\right)\mathfrak{e}_{r\lambda}\right]\Bigg|_{3/2,\widetilde{\rho}_K}\gamma \quad dxdy.\end{aligned}$$

2.6.4. The Shintani theta function

Now we define the theta kernel of the Shintani lift. Recall that for a lattice element $\lambda \in L'/L$ we write $\lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix}$. Let

$$\varphi_{\mathrm{Sh},\Delta}(\lambda, \tau, z) = -\frac{cN\bar{z}^2 - b\bar{z} + a}{4Ny^2}e^{-2\pi v R(\lambda,z)/|\Delta|}e^{2\pi i Q_\Delta(\lambda)\tau}.$$

The Shintani theta function transforms as follows.

Theorem 2.6.12. *The Shintani theta function*

$$\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}}) = v^{1/2}\sum_{h\in\mathcal{D}}\sum_{\substack{\delta\in\mathcal{D}(\Delta)\\ \pi(\delta)=rh\\ Q_\Delta(\delta)\equiv\mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})}}\chi_\Delta(\delta)\sum_{\lambda\in\Delta L+\delta}\varphi_{\mathrm{Sh},\Delta}(\lambda,\tau,z)\mathfrak{e}_h \qquad (2.6.8)$$

is a non-holomorphic automorphic form of weight 2 for $\Gamma_0(N)$ in the variable $z \in D$. Moreover, $\overline{\Theta_{\Delta,r,h}(\tau, z, \varphi_{\mathrm{Sh}})}$ is a non-holomorphic $\mathbb{C}[\mathcal{D}]$-valued modular form of weight $3/2$ for the representation $\widetilde{\rho}$ in the variable τ.

Proof. The automorphic and analytic properties of the untwisted function are stated in [BvdGHZ08] on p. 142. Note that we already explicitly evaluated the function $\frac{(\lambda,\bar{Z})}{(\bar{Z},Z)}$ in our definition above. Using the methods described in Section 2.4 we obtain the desired result. □

2.6.5. Differential equations for theta functions

The theta functions we just defined satisfy some interesting differential equations. These were already investigated in [AGOR, Bru02, BF04, Höv12, BKV13].

The Kudla-Millson theta function and the Siegel theta function are related by the identity [BF04, Theorem 4.4]

$$L_{3/2,\tau}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}) = \frac{1}{4\pi}\Delta_{0,z}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{S}})\cdot\Omega. \tag{2.6.9}$$

For the Kudla-Millson theta kernel we have [BO13, Equation (2.18)]

$$\Delta_{3/2,\tau}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}) = \frac{1}{4}\Delta_{0,z}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}). \tag{2.6.10}$$

The Millson theta function satisfies [Höv12, Proposition 3.10]

$$\Delta_{1/2,\tau}\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}}) = \frac{1}{4}\Delta_{0,z}\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}}). \tag{2.6.11}$$

The Millson and the Shintani theta function are related by the following identity [AGOR, Lemma 3.4]

$$\xi_{1/2,\tau}\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}}) = -4i\sqrt{N}y^2\frac{\partial}{\partial z}\overline{\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{Sh}})}. \tag{2.6.12}$$

All of these identities can be checked by a direct computation. The following identities are essential

$$\begin{aligned}
&\frac{\partial}{\partial z}y^{-2}(cNz^2-bz+a) = -i\sqrt{N}y^{-2}p_z(\lambda),\\
&\frac{\partial}{\partial z}R(\lambda,z) = -\frac{i}{2\sqrt{N}}y^{-2}p_z(\lambda)(cN\bar{z}^2-b\bar{z}+a),\\
&y^{-2}(cNz^2-bz+a)(cN\bar{z}^2-b\bar{z}+a) = 2NR(\lambda,z).
\end{aligned}$$

2.7. Twisted Heegner divisors and the modular trace function

Recall that we defined $L_{m,h}$ for $m\in\mathbb{Q}$ and $h\in L'/L$ by

$$L_{m,h} = \{\lambda\in L+h\,:\,Q(\lambda)=m\}.$$

Moreover, we considered the Heegner divisor $Z(m,h)$ in Section 1.2.4 given by

$$Z(m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L_{m,h}}\frac{1}{|\overline{\Gamma}_\lambda|}Z(\lambda)\in\mathrm{Div}(M)_{\mathbb{Q}},$$

where $\Delta \in \mathbb{Z}$ is a fundamental discriminant and $r \in \mathbb{Z}$ is such that $r^2 \equiv \Delta \pmod{4N}$.

We define a twisted Heegner divisor $Z_{\Delta,r}(m,h)$ by

$$Z_{\Delta,r}(m,h) = \sum_{\lambda \in \Gamma_0(N) \backslash L_{m|\Delta|,rh}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|} Z(\lambda) \in \mathrm{Div}(M)_{\mathbb{Q}}.$$

Here, χ_Δ is the genus character defined in Section 2.4. By Lemma 5.1 of [BO10] $Z_{\Delta,r}(m,h)$ is defined over $\mathbb{Q}(\sqrt{\Delta})$.

Moreover, we let

$$L^+_{|\Delta|m,rh} = \left\{ \lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \in L_{|\Delta|m,rh} \,:\, a \geq 0 \right\},$$

and similarly

$$L^-_{|\Delta|m,rh} = \left\{ \lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \in L_{|\Delta|m,rh} \,:\, a < 0 \right\}.$$

Obviously, we have $L_{|\Delta|m,rh} = L^+_{|\Delta|m,rh} \cup L^-_{|\Delta|m,rh}$.

We define $Z^+_{\Delta,r}(m,h)$ and $Z^-_{\Delta,r}(m,h)$ correspondingly. Similarly as in Lemma 5.1 of [BO10] one can show that $\widetilde{Z}_{\Delta,r}(m,h) = Z^+_{\Delta,r}(m,h) - Z^-_{\Delta,r}(m,h)$ is defined over $\mathbb{Q}(\sqrt{\Delta}, \sqrt{m})$.

Let $k \geq 0$ and let F be a harmonic Maass form of weight $-2k$ for $\Gamma_0(N)$ in $H^+_{-2k}(N)$. We set $\partial F := R^k_{-2k}(F)$ which is automorphic of weight 0.

Definition 2.7.1. If $m \in \mathbb{Q}_{>0}$ with $m \equiv \mathrm{sgn}(\Delta) Q(h)$ $(\mathbb{Z})$ and $h \in \mathcal{D}$ we define the following *modular trace functions*

$$\begin{aligned}
\mathbf{t}^+_{\Delta,r}(F;m,h) &= \sum_{z \in Z^+_{\Delta,r}(m,h)} F(z) = \sum_{\lambda \in \Gamma_0(N) \backslash L^+_{|\Delta|m,rh}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|} \partial F(D_\lambda) \\
\mathbf{t}^-_{\Delta,r}(F;m,h) &= \sum_{\lambda \in \Gamma_0(N) \backslash L^-_{|\Delta|m,rh}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|} \partial F(D_\lambda), \\
\mathbf{t}_{\Delta,r}(F;m,h) &= \sum_{\lambda \in \Gamma_0(N) \backslash L_{|\Delta|m,rh}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|} \partial F(D_\lambda).
\end{aligned}$$

Now let F be a harmonic Maass form of weight 0 in $H^+_0(N)$. We define the "traces" of negative index.

Definition 2.7.2. If $m = 0$ or $m \in \mathbb{Q}_{<0}$ is not of the form $\frac{-Nk^2}{|\Delta|}$ with $k \in \mathbb{Q}_{>0}$ we let

$$\mathbf{t}_{\Delta,r}(F; m, h) = \begin{cases} -\frac{\delta_{h,0}}{2\pi} \int_{\Gamma_0(N)\backslash\mathbb{H}}^{\mathrm{reg}} F(z) \frac{dxdy}{y^2}, & \text{if } \Delta = 1 \\ 0, & \text{if } \Delta \neq 1. \end{cases}$$

Here the integral has to be regularized [BF04, Equation (4.6)].
Now let $m = -Nk^2/|\Delta|$ with $k \in \mathbb{Q}_{>0}$ and $\lambda \in L_{m|\Delta|,rh}$. We have $Q(\lambda) = -Nk^2$, which implies that $\lambda^\perp$ is split over $\mathbb{Q}$ and $c(\lambda)$ is an infinite geodesic. Choose an orientation of V such that

$$\sigma_{\ell_\lambda}^{-1}\lambda = \begin{pmatrix} m & s \\ 0 & -m \end{pmatrix}$$

for some $s \in \mathbb{Q}$. Then c_λ is explicitly given by

$$c_\lambda = \sigma_{\ell_\lambda} \left\{ z \in \mathbb{H} \, : \, \Re(z) = -s/2m \right\}.$$

Define the real part of $c(\lambda)$ by $\Re(c(\lambda)) = -s/2m$. For a cusp ℓ_λ let

$$\langle F, c(\lambda) \rangle = - \sum_{n \in \mathbb{Q}_{<0}} a_{\ell_\lambda}^+(n) e^{2\pi i \Re(c(\lambda))n} - \sum_{n \in \mathbb{Q}_{<0}} a_{\ell_{-\lambda}}^+(n) e^{2\pi i \Re(c(-\lambda))n},$$

where $a_{\ell_\lambda}^+(n)$ denotes the corresponding Fourier coefficient of F at the cusp ℓ. Then we define

$$\mathbf{t}_{\Delta,r}(F; m, h) = \sum_{\lambda \in \Gamma_0(N)\backslash L_{rh,|\Delta|m}} \chi_\Delta(\lambda) \langle F, c(\lambda) \rangle. \tag{2.7.1}$$

The modular trace functions and the traces for the geodesics will appear in the Fourier expansion of the theta lifts that we compute in the following chapters.

3. The Kudla-Millson theta lift

In this chapter we define the Kudla-Millson theta lift. We use the twisted Kudla-Millson theta function defined in the previous chapter as an integration kernel to lift harmonic Maass forms of integer weight $-2k < 0$ to harmonic Maass forms of half-integer weight.

This lift was first considered by Bruinier and Funke in [BF04, BF06]. In [BF06] they showed that the holomorphic part of the lift of a harmonic Maass form F of weight 0 is the generating series for the traces of CM values of F. Ehlen and the author [AE13] later considered a twisted version of this lift which is then a generating series for the twisted traces of CM values of F. Using the Maass lowering and raising operators Bruinier and Ono [BO13] explained a modification of the lift that takes harmonic weak Maass forms of negative weight $-2k$ as an input. They explicitly worked out the lift of weight -2 harmonic Maass forms to obtain algebraic formulas for the partition function $p(n)$. Here, we extend the Kudla-Millson lift to other weights and we include twisted traces.

In the first part of the chapter, we investigate the automorphic and analytic properties of the Kudla-Millson lift. Let $F \in H^{+}_{-2k}(N)$ be a harmonic Maass form of negative weight $-2k$ for $\Gamma_0(N)$. Recall that $\widetilde{\rho} = \rho$ if $\Delta > 0$ and $\widetilde{\rho} = \overline{\rho}$ if $\Delta < 0$. In the case that k is even, the lift of F is a weakly holomorphic modular form of weight $3/2 + k$ for $\widetilde{\rho}$. In the case that k is odd, the lift of F is a harmonic Maass form of weight $1/2 - k$ for $\widetilde{\rho}$ in $H^{+}_{1/2-k,\widetilde{\rho}}$. In this case, the lift is weakly holomorphic if and only if the twisted L-function of $\xi_{-2k}(F)$ vanishes at $s = k + 1$. This gives an interesting new criterion for the vanishing of the L-function in the critical point.

In the second part of the chapter we compute the Fourier expansion of the holomorphic part of the Kudla-Millson lift using a method developed by Katok and Sarnak [KS93]. It turns out that the coefficients of positive index of the holomorphic part are given by the traces of the input function as defined in Section 2.7. We will present applications of this result in Chapter 7.

Throughout the chapter we assume the notation of Section 1.2. In particular, V is a rational quadratic space of signature $(1, 2)$ that we identify with the 2×2 matrices in $\mathbb{Q}$ with trace 0. Recall that M is the modular curve $Y_0(N) = \Gamma_0(N) \setminus \mathbb{H}$. We frequently use the identification between the symmetric space D and the complex upper half plane $\mathbb{H}$ as in (1.2.2). As before, z is used as a variable for integer weight forms (corresponding to automorphic forms on the Grassmannian D), and τ is the symplectic variable that is used for half-integer weight forms. Recall that we write $q = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. Let L be the lattice defined in Section 1.2.1, let $\Delta \in \mathbb{Z}$ be a fundamental discriminant, and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. By ρ we denote the Weil representation associated to the lattice L.

3.1. Definition of the Kudla-Millson theta lift

We let $\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}})$ be the twisted Kudla-Millson function of Section 2.6.3. Recall that it is a non-holomorphic $\mathbb{C}[L'/L]$-valued modular form of weight $3/2$ for the representation $\widetilde{\rho}$ in the variable τ and is $\Gamma_0(N)$-invariant in the variable $z \in D$.

Following Bruinier and Ono [BO13] we define the Kudla-Millson lift of a harmonic Maass form as follows.

Definition 3.1.1. Let $k \geq 0$ be an integer and let F be a harmonic weak Maass form in $H^+_{-2k}(N)$. For even k we define the *Kudla-Millson theta lift* by

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F) = R^{k/2}_{3/2,\tau} \int_M (R^k_{-2k,z}F)(z)\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}}) \tag{3.1.1}$$

and for k odd by

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F) = L^{(k+1)/2}_{3/2,\tau} \int_M (R^k_{-2k,z}F)(z)\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}}). \tag{3.1.2}$$

Note that the rapid decay of the Kudla-Millson function (Proposition 2.6.9) implies that the integrals in (3.1.1) and (3.1.2) exist. In the following section we investigate the automorphic and analytic properties of the lift.

3.2. Automorphic and analytic properties

In this section we investigate the growth of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ at the cusps and its behavior under the Petersson slash operator and the Laplace operator. We obtain the following result.

Theorem 3.2.1. *Let $k > 0$ be an integer and let N be square-free. Let $F \in H^+_{-2k}(N)$ be a harmonic Maass form of weight $-2k$ for $\Gamma_0(N)$. The Kudla-Millson theta lift of F has the following properties:*

(i) If k is odd, the Kudla-Millson theta lift $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ of F is a harmonic weak Maass form of weight $1/2 - k$ transforming with respect to the representation $\widetilde{\rho}$. Moreover, the lift $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ is a weakly holomorphic modular form if and only if the twisted L-function of $\xi_{-2k}(F) \in S_{3/2+k}(N)$ vanishes at $s = k + 1$.

(ii) If k is even, the Kudla-Millson theta lift $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ of F is a weakly holomorphic modular form of weight $3/2 + k$ transforming with respect to the representation $\widetilde{\rho}$.

The case $k = 0$ was treated by Bruinier and Funke [BF06] for $\Delta = 1$. Using the methods presented in Section 2.4 Stephan Ehlen and the author generalized this result to arbitrary fundamental discriminants.

Theorem 3.2.2 (Theorem 4.5, Corollary 4.8 in [BF06], Theorem 5.5, Corollary 5.6 in [AE13])**.** *Let F be a harmonic Maass form of weight 0 for $\Gamma_0(N)$ with vanishing constant* *coefficient at all cusps. The Kudla-Millson theta lift* $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ *of* F *is a weakly holomorphic modular form of weight* $3/2$ *transforming with respect to the representation* $\widetilde{\rho}$*. If the constant coefficients of* F *at all cusps do not vanish, the lift* $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ *lies in the space* $H_{3/2,\widetilde{\rho}}$.

To prove Theorem 3.2.1 we establish a series of results. Note that the transformation properties of the twisted Kudla-Millson theta function $\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}})$ directly imply that the lift transforms with respect to the representation $\widetilde{\rho}$. We first consider the behavior of the lift $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ regarding the weighted Laplace operator. After that we show that the lift also satisfies the correct growth conditions at the cusps under the assumption that N is square-free. We do this by computing the lift of Poincaré series and using the equivariance of the lift with respect to the action of the Atkin-Lehner involutions. Then we investigate under which assumptions the lift is weakly holomorphic.

Proposition 3.2.3. *Let* F *be an eigenform of* $\Delta_{-2k,z}$ *with eigenvalue* λ*. Then* $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ *is an eigenform of* $\Delta_{1/2-k,\tau}$ *with eigenvalue* $\frac{\lambda}{4}$ *if* k *is odd, and* $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ *is an eigenform of* $\Delta_{3/2+k,\tau}$ *with eigenvalue* $\frac{\lambda}{4}$ *if* k *is even.*

Proof. We prove the proposition for odd k. The proof for even k follows analogously. Using Lemma 2.3.9 we see that

$$\begin{aligned}\Delta_{1/2-k,\tau}&\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)\\ &= L^{(k+1)/2}_{3/2,\tau}\int_M (R^k_{-2k,z}F)(z)\Delta_{3/2,\tau}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}) + \frac{k}{4}(k+1)\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F). \qquad (3.2.1)\end{aligned}$$

By equation (2.6.10) we have $\Delta_{3/2,\tau}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}) = \frac{1}{4}\Delta_{0,z}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}})$, which implies that (3.2.1) equals

$$\frac{1}{4}L^{(k+1)/2}_{3/2,\tau}\int_M (R^k_{-2k,z}F)(z)\Delta_{0,z}\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}) + \frac{k}{4}(k+1)\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F).$$

By the rapid decay of the Kudla-Millson theta function (Proposition 2.6.9) we may move the Laplacian. Using Lemma 2.3.9 we then obtain that

$$\Delta_{1/2-k,\tau}\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F) = \frac{1}{4}L^{(k+1)/2}_{3/2,\tau}\int_M (R^k_{-2k,z}\Delta_{-2k,z}F)(z)\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}).$$

Since F is an eigenform of $\Delta_{-2k,z}$ with eigenvalue λ this equals $\frac{\lambda}{4}\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$. ☐

We now compute the lift of the Poincaré series $F_m(z,s,-2k)$. Recall that $\epsilon = 1$ if $\Delta > 0$ and $\epsilon = i$ if $\Delta < 0$.

Theorem 3.2.4. *For $k \geq 0$ even we have*

$$\begin{aligned}
&\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F_m(z,s,-2k)) \\
&\qquad = C^e \cdot \sum_{n|m} \left(\frac{\Delta}{n}\right) n^{-(k+1)} \mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|, -\frac{m}{n}r}\left(\tau, \frac{s}{2}+\frac{1}{4}, \frac{3}{2}+k\right),
\end{aligned}$$

where

$$\begin{aligned}
C^e = -\frac{2^{2s+2k-1} m^{2k+1} \pi^{(3k-1)/2} |\Delta|^{(k+1)/2} \bar{\epsilon}}{N^{k/2}} & \frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma(2s)} \\
& \times \prod_{j=0}^{k-1}(s+j-k) \prod_{j=0}^{k/2-1}\left(\frac{s}{2}+1+j\right),
\end{aligned}$$

and for $k > 0$ odd we have that

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F_m(z,s,-2k)) = C^o \cdot \sum_{n|m} \left(\frac{\Delta}{n}\right) n^{k} \mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|, -\frac{m}{n}r}\left(\tau, \frac{s}{2}+\frac{1}{4}, \frac{1}{2}-k\right),$$

where

$$C^o = -\frac{2^{2k-s}|\Delta|^{-k/2}\bar{\epsilon}}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} N^{(k+1)/2}\pi^{k/2} s \prod_{j=0}^{k-1}(s+j-k) \prod_{j=0}^{(k-1)/2}\left(\frac{s}{2}-\frac{1}{2}-j\right).$$

Proof. For the explicit evaluation of the lift of Poincaré series we generalize the proof of Bruinier and Ono [BO13]. Repeatedly applying Proposition 2.5.1 implies by induction

$$\begin{aligned}
&\int_M (R^k_{-2k,z} F_m(z,s,-2k)) \Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}) \\
&= (4\pi m)^k \prod_{j=0}^{k-1}(s+j-k) \int_M F_m(z,s,0)\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}).
\end{aligned} \tag{3.2.2}$$

Using the definition of the Poincaré series (2.5.1) and an unfolding argument we obtain

$$\frac{1}{\Gamma(2s)}(4\pi m)^k \prod_{j=0}^{k-1}(s+j-k) \int_{\Gamma_\infty \backslash \mathbb{H}} \mathcal{M}_{s,0}(4\pi m y) e(-mx) \Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}}).$$

By Proposition 2.6.11 this equals

$$-\frac{N^{3/2}\bar{\epsilon}}{2|\Delta|}\frac{(4\pi m)^k}{\Gamma(2s)} \prod_{j=0}^{k-1}(s+j-k) \sum_{n=1}^{\infty}\left(\frac{\Delta}{n}\right) n^2 \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} I(\tau,s,m,n)|_{3/2,\widetilde{\rho}_K}\, \gamma,$$

where

$$I(\tau,s,m,n)=\int_{y=0}^{\infty}\int_{x=0}^{1} y\mathcal{M}_{s,0}(4\pi my)e(-mx)\exp\left(-\frac{\pi n^2Ny^2}{|\Delta|\,v}\right)$$
$$\times\, v^{-3/2}\sum_{\lambda\in K'} e\left(|\Delta|\,Q(\lambda)\bar\tau-2N\lambda nx\right)\mathfrak{e}_{r\lambda}dxdy.$$

Identifying $K'=\mathbb{Z}\left(\begin{smallmatrix}1/2N & 0\\ 0 & -1/2N\end{smallmatrix}\right)$ we find that

$$\sum_{\lambda\in K'} e\left(|\Delta|\,Q(\lambda)\bar\tau-2N\lambda nx\right)\mathfrak{e}_{r\lambda}=\sum_{b\in\mathbb{Z}} e\left(-|\Delta|\,\frac{b^2}{4N}\bar\tau-nbx\right)\mathfrak{e}_{rb}.$$

Inserting this in the formula for $I(\tau,s,m,n)$ and integrating over x, we see that $I(\tau,s,m,n)$ vanishes whenever $n\nmid m$ and the only summand occurs for $b=-m/n$ when $n\mid m$. Thus, $I(\tau,s,m,n)$ equals

$$v^{-3/2}e\left(-|\Delta|\,\frac{m^2}{4Nn^2}\bar\tau\right)\cdot\int_{y=0}^{\infty} y\mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi n^2Ny^2}{|\Delta|\,v}\right)dy\;\mathfrak{e}_{-rm/n}. \tag{3.2.3}$$

To evaluate the integral in (3.2.3) note that (see for example (13.6.3) in [AS84])

$$\mathcal{M}_{s,0}(4\pi my)=2^{2s-1}\Gamma\left(s+\frac{1}{2}\right)\sqrt{4\pi my}\cdot I_{s-1/2}(2\pi my).$$

Substituting $t=y^2$ yields

$$\begin{aligned}&\int_{y=0}^{\infty} y\mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi n^2Ny^2}{|\Delta|\,v}\right)dy\\&=2^{2s-1}\Gamma\left(s+\frac{1}{2}\right)\int_{y=0}^{\infty} y\sqrt{4\pi my}\;I_{s-1/2}(2\pi my)\exp\left(-\frac{\pi n^2Ny^2}{|\Delta|\,v}\right)dy\\&=2^{2s-1}\Gamma\left(s+\frac{1}{2}\right)\sqrt{m\pi}\int_{t=0}^{\infty} t^{1/4}I_{s-1/2}(2\pi mt^{1/2})\exp\left(-\frac{\pi n^2Nt}{|\Delta|\,v}\right)dt.\end{aligned}$$

The last integral is a Laplace transform and is computed in [EMOT54] (see (20) on p. 197). It equals

$$\frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(s+\frac{1}{2}\right)}(\pi m)^{-1}\left(\frac{\pi n^2N}{|\Delta|\,v}\right)^{-3/4}\exp\left(\frac{\pi m^2\,|\Delta|\,v}{2n^2N}\right)M_{-\frac{3}{4},\frac{s}{2}-\frac{1}{4}}\left(\frac{\pi m^2\,|\Delta|\,v}{n^2N}\right).$$

Inserting this and using that $\mathcal{M}_{s,k}(y) = y^{-k/2} M_{-\frac{k}{2}, s-\frac{1}{2}}(y)$ we obtain that

$$I(\tau, s, m, n) = 2^{2s-1} \Gamma\left(\frac{s}{2}+1\right)(\pi m)^{-2}\left(\frac{\pi m^2 |\Delta|}{n^2 N}\right)^{3/2} \times \mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}}\left(\frac{\pi m^2 |\Delta| v}{n^2 N}\right) e\left(-\frac{m^2 |\Delta| u}{4n^2 N}\right) \mathfrak{e}_{-rm/n}.$$

Therefore, we have that (3.2.2) equals

$$-\frac{N^{3/2}}{2|\Delta|}\bar{\epsilon}\,\frac{1}{\Gamma(2s)}(4\pi m)^k \prod_{j=0}^{k-1}(s+j-k) \sum_{n|m}\left(\frac{\Delta}{n}\right) n^2 \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} I(\tau, s, m, n)|_{3/2, \widetilde{\rho}_K}\gamma. \tag{3.2.4}$$

For $k \geq 0$ even we write for equation (3.2.4)

$$C \cdot \frac{1}{2}\sum_{n|m}\left(\frac{\Delta}{n}\right) n^{-1} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})}\left[\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}}\left(\frac{\pi m^2 |\Delta| v}{n^2 N}\right) e\left(-\frac{m^2|\Delta| u}{4n^2 N}\right)\mathfrak{e}_{-rm/n}\right]\Bigg|_{3/2,\widetilde{\rho}_K}\gamma$$

with

$$C = -2^{2s+2k-1}\, m^{k+1}\, \pi^{k-1/2}\sqrt{|\Delta|}\,\bar{\epsilon}\,\frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma(2s)}\prod_{j=0}^{k-1}(s+j-k).$$

We now apply the differential operator $R^{k/2}_{3/2,\tau}$ to this expression. By the commutativity of the raising and the slash operator, Proposition 2.5.1 implies

$$R^{k/2}_{3/2,\tau}\left(\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}}\left(\frac{\pi m^2|\Delta| v}{n^2 N}\right) e\left(-\frac{m^2|\Delta| u}{4n^2 N}\right)\right) = \left(4\pi\frac{m^2|\Delta|}{4Nn^2}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{s}{2}+\frac{1}{4}+\frac{3/2+2j}{2}\right)\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}}\left(\frac{\pi m^2|\Delta| v}{n^2 N}\right) e\left(-\frac{m^2|\Delta| u}{4n^2 N}\right).$$

We collect terms to obtain C^{e} as in the statement of the theorem.

For odd k we rewrite (3.2.4) as follows

$$C' \cdot \frac{1}{2\Gamma\left(s+\frac{1}{2}\right)}\sum_{n|m}\left(\frac{\Delta}{n}\right) n^{-1} \times \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})}\left[\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}}\left(\frac{\pi m^2|\Delta| v}{n^2 N}\right) e\left(-\frac{m^2|\Delta| u}{4n^2 N}\right)\mathfrak{e}_{-rm/n}\right]\Bigg|_{3/2,\widetilde{\rho}_K}\gamma$$

where

$$C' := -\frac{2^{2k-s}\sqrt{|\Delta|}\,\bar{\epsilon}}{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}\, m^{k+1}\,\pi^{k+1/2} s\prod_{j=0}^{k-1}(s+j-k).$$

A repeated application of the lowering operator on $\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}}\left(\frac{\pi m^2|\Delta|v}{n^2N}\right)e\left(-\frac{m^2|\Delta|u}{4n^2N}\right)$ as in Lemma 2.5.2 yields the statement in the theorem since the lowering and slash operator commute. $\square$

We now investigate the operation of the Atkin-Lehner involutions on the Kudla-Millson lift.

Proposition 3.2.5. *For an Atkin-Lehner involution W_Q^N as in Definition 1.2.2, $h \in L'/L$ and $F \in H_{-2k}(N)$, we have*

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r,W_Q^N.h}(\tau,F) = \mathcal{I}^{\mathrm{KM}}_{\Delta,r,h}\left(\tau, F|_{-2k}(W_Q^N)^{-1}\right).$$

Proof. This follows from Proposition 2.6.10. $\square$

Using the action of the Atkin-Lehner involutions and the fact that the Poincaré series generate the space of harmonic Maass forms of weight less than zero, we are able to prove that the Kudla-Millson lifts of harmonic Maass forms are again harmonic Maass forms.

Corollary 3.2.6. *Let N be square-free and $k > 0$. If $F \in H^+_{-2k}(N)$ is a harmonic Maass form of weight $-2k$ for $\Gamma_0(N)$, then $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ belongs to $M^!_{3/2+k,\widetilde{\rho}}$ if k is even and $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ belongs to $H^+_{1/2-k,\widetilde{\rho}}$ if k is odd.*

Proof. For $f \in H^+_{3/2+k,\widetilde{\rho}}$, where $k > 0$, we have $\xi_{3/2+k}(f) \in S_{1/2-k,\bar{\widetilde{\rho}}}$. Since $\dim(S_{1/2-k,\bar{\widetilde{\rho}}}) = 0$ for $k > 0$, this implies that $f \in M^!_{3/2+k,\widetilde{\rho}}$.

For odd k the proof is similar to the proof of [BO13, Corollary 3.4]: For $m \in \mathbb{Z}_{>0}$ the Poincaré series span the subspace $H^{+,\infty}_{-2k}(N)$ of harmonic Maass forms whose principal parts at all cusps other than ∞ are constant. By Theorem 3.2.4 we find that the image of $H^{+,\infty}_{-2k}(N)$ is contained in $M^!_{3/2+k,\widetilde{\rho}}$ for even k, and $H^+_{1/2-k,\widetilde{\rho}}$ for odd k. Let W denote the group of Atkin-Lehner involutions. Then,

$$H^+_{-2k}(N) = \sum_{\gamma\in W}\gamma H^{+,\infty}_{-2k}(N),$$

since the group W acts transitively on the cusps of $\Gamma_0(N)$ for square-free N. Applying Proposition 3.2.5 now implies the result. $\square$

A natural question is under which conditions the Kudla-Millson lift of $F \in H^+_{-2k}(N)$ with k odd is weakly holomorphic. We obtain the following interesting criterion that relates the weak holomorphicity of the lift to the vanishing of the twisted L-function of $\xi_{-2k}(F)$ at the critical point.

Theorem 3.2.7. *If k is odd, N is square-free, and $F \in H^+_{-2k}(N)$, then $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ is weakly holomorphic if and only if we have*

$$L(\xi_{-2k}(F),\Delta,k+1) = 0.$$

In particular, this is true when F is weakly holomorphic.

Proof. Here we partly follow the proof of Bruinier and Ono in [BO13]. Since the Atkin-Lehner involutions act transitively on the cusps, it suffices to consider the case when the principal parts of F at all cusps other than ∞ are constant. Again, we obtain the result for the entire space $H^+_{-2k}(N)$ by using Proposition 3.2.5. For $F \in H^{+,\infty}_{-2k}(N)$ we denote the Fourier expansion of the holomorphic part at the cusp ∞ by

$$F(z) = \sum_{m \in \mathbb{Z}} a_F(m) e(mz).$$

Then we can write F as a linear combination of Poincaré series

$$F(z) = \sum_{m>0} a_F(-m) F_m(z, 1+k, -2k).$$

By Theorem 3.2.4 the principal part of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ is given by

$$C^{\mathrm{o}} \cdot \sum_{m>0} a_F(-m) \sum_{n|m} \left(\frac{\Delta}{n}\right) n^k e\left(-\frac{m^2 |\Delta|}{4Nn^2} z\right) (\mathfrak{e}_{rm/n} + \mathfrak{e}_{-rm/n}),$$

where C^{o} is as in Theorem 3.2.4.

We use the pairing between the spaces $H^+_{1/2-k,\widetilde{\rho}}$ and $S_{3/2+k,\bar{\widetilde{\rho}}}$ (see Equation (2.3.9)). To prove that the lift is weakly holomorphic we have to show that $\left\{g, \mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)\right\} = 0$ for every cusp form $g \in S_{3/2+k,\bar{\widetilde{\rho}}}$. Denoting the Fourier coefficients of g by $b(n,h)$, we have

$$\begin{aligned} \left\{g, \mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)\right\} &= 2C^{\mathrm{o}} \sum_{m>0} a_F(-m) \sum_{n|m} n^k \left(\frac{\Delta}{n}\right) b\left(\frac{m^2 |\Delta|}{4Nn^2}, \frac{m}{n} r\right) \\ &= 2C^{\mathrm{o}} \left\{F, \mathcal{S}_{\Delta,r}(g)\right\} = 2C^{\mathrm{o}} (\xi_{-2k}(F), \mathcal{S}_{\Delta,r}(g))_{2k+2}, \end{aligned}$$

where $\mathcal{S}_{\Delta,r}(g) \in S_{2k+2}(N)$ denotes the Shimura lift of g as in [Sko90a]. If F is weakly holomorphic this expression vanishes, since $\xi_{-2k}(F) = 0$.

If $F \in H^+_{-2k}(N) \setminus M^!_{-2k}(N)$, we have by the adjointness of the Shintani and Shimura lift (see for example Section II.3 of [GKZ87], and [Sko90a, Sko90b] for the case of skew-holomorphic Jacobi forms)

$$(\xi_{-2k}(F), \mathcal{S}_{\Delta,r}(g))_{2k+2} = (\mathcal{S}^*_{\Delta,r}(\xi_{-2k}(F)), g)_{3/2+k},$$

where $S^*_{\Delta,r}$ denotes the Shintani lift (this notation differs from the one we will use in the next chapter, but is consistent with the notation in [GKZ87]). This equals zero for all cusp forms g if and only if the Shintani lift of $\xi_{-2k}(F)$ vanishes. In terms of Jacobi forms we have that

$$\mathcal{S}^*_{\Delta,r}(\xi_{-2k}(F)) = \left(\frac{i}{2N}\right)^k \sum_{\substack{n,r_0\in\mathbb{Z}\\ r_0^2<4nN}} r_{k+1,N,\Delta(r_0^2-4nN),rr_0,\Delta}(\xi_{-2k}(F))q^n\zeta^{r_0}, \tag{3.2.5}$$

where $r_{k+1,N,\Delta(r_0^2-4nN),rr_0,\Delta}$ is a certain cycle integral defined in [GKZ87].

Now by the Theorem and Corollary in Section II.4 in [GKZ87] we have

$$\begin{aligned}\left|r_{k+1,N,\Delta(r_0^2-4nN),rr_0,\Delta}(\xi_{-2k}(F))\right|^2 &= |\Delta|^{k+1/2}\left|r_0^2-4nN\right|^{k+1/2}(k!)^2N^{-k}\\ &\times 2^{-3k-2}\pi^{-2(k+1)}L(\xi_{-2k}(F),\Delta,k+1)\cdot L(\xi_{-2k}(F),r_0^2-4nN,k+1).\end{aligned}$$

From Theorem B of [LR97] we know that $L(\xi_{-2k}(F),r_0^2-4nN,k+1)=0$ for all r_0 and n implies that $\xi_{-2k}(F)$ vanishes. Therefore, since r_0 and n vary in (3.2.5) the Shintani lift $\mathcal{S}^*_{\Delta,r}(\xi_{-2k}(F))$ vanishes if and only if $L(\xi_{-2k}(F),\Delta,k+1)$ vanishes. □

3.3. Fourier coefficients of the holomorphic part

Now we turn to the computation of the Fourier coefficients of positive index of the holomorphic part of the theta lift. Recall that the modular trace function is defined as

$$\mathbf{t}_{\Delta,r}(F;m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L_{|\Delta|m,rh}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|}\partial F(D_\lambda),$$

where $\partial F := R^k_{-2k}(F)$ for $F\in H^+_{-2k}(N)$ with $k\geq 0$, $h\in L'/L$ and $m\in\mathbb{Q}_{>0}$ with $m\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})$ (see also Section 2.7).

Theorem 3.3.1. *We let $k\geq 0$ be an integer, $F\in H^+_{-2k}(N)$, $h\in L'/L$, and let $m\in\mathbb{Q}_{>0}$ with $m\equiv\operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})$. We obtain the following results:*

(i) Let k be even. The (m,h)-th Fourier coefficient of the holomorphic part of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ equals

$$\left(-\frac{4\pi m}{|\Delta|}\right)^{k/2}\mathbf{t}_{\Delta,r}(F;m,h).$$

(ii) Let k be odd. The (m,h)-th Fourier coefficient of the holomorphic part of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ equals

$$\left(\frac{|\Delta|}{4\pi m}\right)^{(k+1)/2}\prod_{j=0}^{(k-1)/2}\left(\frac{k}{2}+j\right)\left(j-\frac{k+1}{2}\right)\mathbf{t}_{\Delta,r}(F;m,h).$$

Remark 3.3.2. As described in Section 1.2.1 we can identify the elements of L'/L with binary quadratic forms. Combining work of Miller and Pixton [MP10] and of Bruinier and Ono [BO13] one can show algebraicity results for the numbers $(R^k_{-2k,z}F)(\alpha_Q)$ occurring in the traces above.

To be more precise, let N be square-free and D be a positive discriminant that is coprime to N. Moreover, let $r \in \mathbb{Z}$ with $r^2 \equiv -D \pmod{4N}$. By $\mathcal{Q}_{D,r,N}$ we denote the set of positive definite integral binary quadratic forms $[a,b,c]$ of discriminant $-D$ with $N|a$ and $b \equiv r \pmod{2N}$. Then we let $\alpha_Q = \frac{-b+\sqrt{-D}}{2a}$ be the Heegner point corresponding to $Q \in \mathcal{Q}_{D,r,N}$. We write $\mathcal{O}_D$ for the order of discriminant $-D$ in $\mathbb{Q}(\sqrt{-D})$.

Using the work of Miller and Pixton [MP10] and of Bruinier and Ono [BO13] it is shown in [Alf14, Theorem 6.1]) that for $F \in M^!_{-2k}(N)$ with integral coefficients at all cusps and primitive $Q \in \mathcal{Q}_{D,r,N}$ the number $3^k D^k (\frac{1}{\pi})^k (R^k_{-2k,z}F)(\alpha_Q)$ is an algebraic integer in the ring class field for the order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{-D})$. The multiset of values $R^k_{-2k}f(\alpha_Q)$ is a union of Galois orbits.

Proof of Theorem 3.3.1. We first prove the result for $\Delta = 1$. Using the methods developed in [AE13] we then deduce the general result.
Proof for $\Delta = 1$*:* We consider the Fourier expansion of $\int_M \partial F(z)\Theta(\tau, z, \varphi_{\mathrm{KM}})$, namely

$$\sum_{h\in L'/L}\sum_{m\in\mathbb{Q}}\left(\sum_{\lambda\in L_{m,h}}\int_M \partial F(z)\varphi^0_{\mathrm{KM}}(\sqrt{v}\lambda, z)\right)e^{2\pi i m\tau}. \tag{3.3.1}$$

We denote the (m,h)-th coefficient of the holomorphic part of (3.3.1) by $C(m,h)$. Using the usual unfolding argument we find

$$C(m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|}\int_M \partial F(z)\varphi^0_{\mathrm{KM}}(\sqrt{v}\lambda, z).$$

Following Bruinier and Ono [BO13, Proof of Theorem 3.6] we employ an argument of Katok and Sarnak [KS93]. We rewrite the integral over M as an integral over $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. Here, we normalize the Haar measure such that the maximal compact open subgroup has volume 1. Then we have

$$C(m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|}\int_{G(\mathbb{R})} \partial F(gi)\varphi^0_{\mathrm{KM}}(\sqrt{v}\lambda, gi)dg.$$

Note that $\mathrm{SL}_2(\mathbb{R})$ does not act transitively on elements of the same norm as it is assumed in [KS93]. But when splitting the elements $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N\end{smallmatrix}\right)$ into two subsets, depending on whether $a \geq 0$ or $a < 0$ as described in Section 2.7, we see that $\mathrm{SL}_2(\mathbb{R})$ acts transitively on these subsets. There is a $g_1 \in \mathrm{SL}_2(\mathbb{R})$ such that $g_1.\lambda = \sqrt{m}\lambda(i)$ if $a \geq 0$. If $a < 0$, we have $g_1.(-\lambda) = \sqrt{m}\lambda(i)$ and use that φ^0_{KM} is an even function in the first variable. For the element $g_1 \in \mathrm{SL}_2(\mathbb{R})$ we find

$$g_1 = \begin{pmatrix} \left(\frac{\sqrt{4NQ(\lambda)}}{2cN}\right)^{1/2} & \frac{b}{2cN}\left(\frac{\sqrt{4NQ(\lambda)}}{2cN}\right)^{-1/2} \\ 0 & \left(\frac{\sqrt{4NQ(\lambda)}}{2cN}\right)^{-1/2} \end{pmatrix}.$$

Then, $g_1 gi$ is the Heegner point corresponding to D_λ.

By the invariance of the Haar measure we obtain

$$C(m,h) = \sum_{\lambda \in \Gamma_0(N)\backslash L_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|} \int_{G(\mathbb{R})} \partial F(g_1 gi)\varphi^0_{\mathrm{KM}}\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right) dg.$$

We set

$$I(\lambda) := \int_{G(\mathbb{R})} \partial F(g_1 gi)\varphi^0_{\mathrm{KM}}\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right) dg,$$

and

$$f(g) := \partial F(g_1 gi).$$

Using the Cartan decomposition KA^+K of $\mathrm{SL}_2(\mathbb{R})$ and the K-invariance of the function $\varphi^0_{\mathrm{KM}}\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right)$, we then find as in [KS93] that

$$\begin{aligned} I(\lambda) = 4\pi \int_1^\infty & \varphi^0_{\mathrm{KM}}(\sqrt{mv}\alpha(a)^{-1}\lambda(i), i) \\ & \times \left(\int_K \int_K f(k_1\alpha(a)k_2)dk_1dk_2\right)\frac{a^2 - a^{-2}}{2}\frac{da}{a}. \end{aligned}$$

The function

$$\tilde{f}(g) := \int_K \int_K f(k_1\alpha(a)k_2)dk_1dk_2$$

has the same eigenvalues as $\partial F(gi)$ under the action of the Laplace operator and is right and left K-invariant. By using the uniqueness of spherical functions we obtain

$$\tilde{f}(g) = \tilde{f}(1)\cdot\omega_c(g) = \partial F(D_\lambda)\cdot\omega_c(g)$$

where $\omega_c(g)$ is the standard spherical function of eigenvalue $c = -k(k+1)$. We find

$$I(\lambda) = \partial F(D_\lambda)\cdot Y_c(t),$$

where

$$Y_c(t) = 4\pi \int_1^\infty \varphi^0_{\mathrm{KM}}(t\alpha(a)^{-1}.\lambda(i), i)\omega_c(\alpha(a))\frac{a^2 - a^{-2}}{2}\frac{da}{a}, \tag{3.3.2}$$

and $\alpha(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$. Note that $\omega_c(\alpha(a)) = \omega_c\left(\frac{a^2+a^{-2}}{2}\right)$.

By evaluating $\varphi^0_{\mathrm{KM}}(t\alpha(a)^{-1}.\lambda(i), i)$ using (2.6.1) and (2.6.2) and substituting $a = e^{r/2}$ we obtain that (3.3.2) equals

$$2\pi \int_0^\infty \left(4t^2\cosh(r)^2 - \frac{1}{2\pi}\right)\omega_c(\cosh(r))\sinh(r)e^{-4\pi t^2\sinh(r)^2}dr. \tag{3.3.3}$$

In this case the standard spherical function is given by the Legendre polynomial $P_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}(x^2-1)^k$ [Iwa02, Chapter 1]. By substituting $x = \sinh(r)^2$ we obtain that (3.3.3) equals

$$4\pi t^2 \int_0^\infty \sqrt{1+x}P_k(\sqrt{1+x})e^{-4\pi t^2 x}dx - \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{1+x}}P_k(\sqrt{1+x})e^{-4\pi t^2 x}dx. \tag{3.3.4}$$

To evaluate the first integral in (3.3.4) we use the following recursion formula for the Legendre polynomial (see for example equation (8.5.3) in [AS84])

$$\sqrt{1+x}P_k(\sqrt{1+x}) = \frac{1}{2k+1}\left((k+1)P_{k+1}(\sqrt{1+x}) + kP_{k-1}(\sqrt{1+x})\right).$$

Thus, we are left with

$$4\pi t^2\int_0^\infty \left(\frac{k+1}{2k+1}P_{k+1}(\sqrt{1+x}) + \frac{k}{2k+1}P_{k-1}(\sqrt{1+x})\right)e^{-4\pi t^2 x}dx, \tag{3.3.5}$$

which is a Laplace transform computed in [EMOT54] (see equation (7) on page 180). It equals

$$(4\pi t^2)^{-1/4}e^{2\pi t^2}\left(\frac{k+1}{2k+1}W_{\frac{1}{4},\frac{k}{2}+\frac{3}{4}}(4\pi t^2) + \frac{k}{2k+1}W_{\frac{1}{4},\frac{k}{2}-\frac{1}{4}}(4\pi t^2)\right). \tag{3.3.6}$$

The second integral in (3.3.4) can be evaluated in the same way (see equation (8) on page 180 of [EMOT54]) and equals

$$-\frac{1}{2}(4\pi t^2)^{-3/4}e^{2\pi t^2}W_{-\frac{1}{4},\frac{k}{2}+\frac{1}{4}}(4\pi t^2). \tag{3.3.7}$$

Using (13.1.33), (13.4.17), and (13.4.20) in [AS84] together with Equation (2.5.2) it is not hard to show that the sum of the expressions in (3.3.6) and (3.3.7) is equal to

$$e^{2\pi t^2}\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}}(4\pi t^2).$$

Thus, $C(m,h)$ is given by

$$C(m,h) = \sum_{\lambda\in L_{m,h}}\frac{1}{|\overline{\Gamma}_\lambda|}\partial F(D_\lambda)e^{2\pi m v}\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}}(4\pi m v).$$

We now have to apply the iterated raising respectively lowering operator to the Fourier expansion in (3.3.1), which boils down to evaluating it on

$$q^m e^{2\pi mv}\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}}(4\pi mv) = \mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}}(4\pi mv)e(mx).$$

By Proposition 2.5.4 we obtain

$$\begin{aligned}R^{k/2}_{3/2,\tau}&\left(\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}}(4\pi mv)e(mx)\right)\\&= (-4\pi m)^{k/2}\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}+k}(4\pi mv)e(mx) = (-4\pi m)^{k/2}q^m,\end{aligned}$$

since $W_{\nu,\mu}(y) = y^{k/2}e^{-y/2}$ for $y > 0$, $\nu = k/2$, and $\mu = k/2 - 1/2$ [AS84, Chapter 13].

For the lowering operator a repeated application of Proposition 2.5.4 yields

$$\begin{aligned}&L^{(k+1)/2}_{3/2,\tau}\left(\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{3}{2}}(4\pi mv)e(mx)\right)\\&= \left(\frac{1}{4\pi m}\right)^{(k+1)/2}\prod_{j=0}^{(k-1)/2}\left(\frac{k}{2}+j\right)\left(j-\frac{k+1}{2}\right)\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}-k}(4\pi mv).\end{aligned}$$

Again the Whittaker function simplifies, namely $\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}-k}(4\pi mv) = e^{-2\pi mv}$.

Thus, we obtain that $C(m,h)$ is given by

$$C(m,h) = (-4\pi m)^{k/2}\sum_{\lambda\in L_{m,h}}\frac{1}{|\Gamma_\lambda|}\partial F(D_\lambda) = (-4\pi m)^{k/2}\mathbf{t}(F;m,h), \tag{3.3.8}$$

in the case that the input function F has weight $-2k$ with k even. In the case that k is odd, we obtain

$$C(m,h) = \left(\frac{1}{4\pi m}\right)^{(k+1)/2}\prod_{j=0}^{\frac{k-1}{2}}\left(\frac{k}{2}+j\right)\left(j-\frac{k+1}{2}\right)\mathbf{t}(F;m,h). \tag{3.3.9}$$

Proof for $\Delta \neq 1$: Replacing the theta function $\Theta(\tau,z,\varphi_{\mathrm{KM}})$ by $\Theta_{\Delta,r}(\tau,z,\varphi_{\mathrm{KM}})$ we can write

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F) = \sum_{h\in\mathcal{D}}\left\langle\psi_{\Delta,r}(\mathfrak{e}_h), \int_M \partial F(z)\overline{\Theta_{\mathcal{D}(\Delta)}(\tau,z,\varphi_{\mathrm{KM}})}\right\rangle.$$

In general the group $\Gamma_0(N)$ does not act trivially on $\mathcal{D}(\Delta)$. However, the Kudla-Millson theta function $\Theta_{\mathcal{D}(\Delta)}(\tau,z,\varphi_{\mathrm{KM}})$ is always invariant under the discriminant kernel $\Gamma_\Delta = \{\gamma\in\Gamma_0(N) : \gamma\delta=\delta \text{ for all } \delta\in\mathcal{D}(\Delta)\}\subset\Gamma_0(N)$. Since $\partial F(z)\Theta_{\mathcal{D}(\Delta)}(\tau,z,\varphi_{\mathrm{KM}})$ is $\Gamma_0(N)$-invariant by (2.6.7) we obtain by a standard argument

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F) = \frac{1}{[\Gamma_0(N):\Gamma_\Delta]}\sum_{h\in\mathcal{D}}\left\langle\psi_{\Delta,r}(\mathfrak{e}_h), \int_{\Gamma_\Delta\backslash\mathbb{H}} \partial F(z)\overline{\Theta_{\mathcal{D}(\Delta)}(\tau,z,\varphi_{\mathrm{KM}})}\right\rangle.$$

Now we are able to apply the result for the coefficients in the case $\Delta = 1$ to the integral above. Note that we have to replace m by $m/|\Delta|$ in (3.3.8) and (3.3.9).

For $m \in \mathbb{Q}$ we then obtain that the (m,h)-th Fourier coefficient of the holomorphic part of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ is given by $1/[\Gamma_0(N) : \Gamma_\Delta]$ times

$$\left\langle \psi_{\Delta,r}(\mathfrak{e}_h), \sum_{\delta \in \mathcal{D}(\Delta)} \overline{C(m,h)} \mathfrak{e}_\delta \right\rangle = \sum_{\substack{\delta \in \mathcal{D}(\Delta) \\ \pi(\delta) = rh \\ Q_\Delta(\delta) \equiv \mathrm{sgn}(\Delta) Q(h)\, (\mathbb{Z})}} \chi_\Delta(\delta)\ \mathbf{t}(F; m, \delta) \tag{3.3.10}$$

$$\times \begin{cases} \left(-\frac{4\pi m}{|\Delta|}\right)^{k/2} & \text{if } k \text{ is even,} \\ \left(\frac{|\Delta|}{4\pi m}\right)^{(k+1)/2} \prod_{j=0}^{\frac{k-1}{2}} \left(\frac{k}{2} + j\right)\left(j - \frac{k+1}{2}\right) & \text{if } k \text{ is odd.} \end{cases} \tag{3.3.11}$$

Here, the traces are taken with respect to Γ_Δ and the discriminant group $\mathcal{D}(\Delta)$. Note that

$$\mathbf{t}(F; m, \delta) = \sum_{\lambda \in \Gamma_\Delta \backslash (\Delta L)_{\delta, m}} \frac{1}{\left|\overline{\Gamma}_{\Delta,\lambda}\right|} \partial F(D_\lambda). \tag{3.3.12}$$

where $(\Delta L)_{\delta,m} = \{\lambda \in \Delta L + \delta : Q_\Delta(\lambda) = m\}$. If $m \equiv \mathrm{sgn}(\Delta) Q(h) \pmod{\mathbb{Z}}$ the right hand side in (3.3.10) is equal to $\mathbf{t}_{\Delta,r}(F; m, h)$ and we obtain the result stated in the theorem. □

3.4. Fourier expansion in the case $k = 0$

For the sake of completeness we briefly state the Fourier expansion of the lift in the case $k = 0$. It was derived by Bruinier and Funke in [BF06] for $\Delta = 1$ and by Ehlen and the author in [AE13] for $\Delta \neq 1$.

Theorem 3.4.1 (Theorem 4.5 in [BF06], Theorem 5.5 in [AE13])**.** *Let $F \in H_0^+(N)$ and write*

$$F(\sigma_\ell z) = \sum_{\substack{n \in \frac{1}{\alpha_\ell}\mathbb{Z} \\ n \gg -\infty}} a_\ell^+(n) q^n + \sum_{\substack{n \in \frac{1}{\alpha_\ell}\mathbb{Z} \\ n < 0}} a_\ell^-(n) \bar{q}^n$$

for the Fourier expansion of F at the cusp ℓ. Assume that F has vanishing constant term at every cusp of $\Gamma_0(N)$. Then the Fourier expansion of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r,h}(\tau, F)$ is given by

$$\mathcal{I}^{\mathrm{KM}}_{\Delta,r,h}(\tau, F) = \sum_{\substack{m \in \mathbb{Q}_{>0} \\ m \equiv \mathrm{sgn}(\Delta) Q(h)\, (\mathbb{Z})}} \mathbf{t}_{\Delta,r}(F; m, h) q^m + \sum_{\substack{m \in \mathbb{Q}_{>0} \\ -N|\Delta| m^2 \equiv \mathrm{sgn}(\Delta) Q(h)\, (\mathbb{Z})}} \mathbf{t}_{\Delta,r}(F; -N|\Delta| m^2, h) q^{-N|\Delta| m^2}.$$

If the constant coefficients of F at the cusps do not vanish, the following terms occur in *addition:*

$$\frac{\sqrt{|\Delta|}}{2\pi\sqrt{Nv}} \sum_{\ell \in \Gamma_0(N)\backslash \mathrm{Iso}(V)} \mathfrak{d}_{\Delta,r}(\ell,h)\, \epsilon_\ell\, a_\ell^+(0)$$
$$+\sqrt{|\Delta|}\sum_{m>0} \sum_{\substack{\lambda \in L_{rh,-Nm^2} \\ Q_\Delta(\lambda) \equiv \mathrm{sgn}(\Delta) Q(h)\,(\mathbb{Z})}} \chi_\Delta(\lambda) \frac{a_{\ell_\lambda}^+(0) + a_{\ell_{-\lambda}}^+(0)}{8\pi\sqrt{vN}m} \beta\left(\frac{4\pi v N m^2}{|\Delta|}\right) q^{-Nm^2/|\Delta|},$$

where $\beta(s) = \int_1^\infty t^{-3/2} e^{-st} dt$ *and* $\mathfrak{d}$ *is defined in Remark 3.4.3.*

Remark 3.4.2. The coefficients of the principal part of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ can be computed explicitly in terms of the coefficients of the principal part of F (see [BF06, Proposition 4.7] and [AE13, Proposition 5.7]).

Remark 3.4.3. We briefly explain the nature of the constant $\mathfrak{d}$ as defined in [AE13]. For $h \in L'/L$ and $\ell \in \mathrm{Iso}(V)$, we let

$$\delta_\ell(h) = \begin{cases} 1, & \text{if } \ell \cap (L+h) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.1}$$

If $\delta_\ell(h) = 1$, there is an h_ℓ such that $\ell \cap (L+h) = \mathbb{Z}\lambda_\ell + h_\ell$. Now let $s \in \mathbb{Q}$ such that $h_\ell = s\lambda_\ell$. Write $s = \frac{p}{q}$ with $(p, q) = 1$ and define $d(\ell, h) := q$, which depends only on ℓ and h. Moreover, we define $h'_\ell = \frac{1}{d(\ell,h)}\lambda_\ell$ which is well defined as an element of $\mathcal{D}$.

Then $\mathfrak{d}_{\Delta,r}(\ell, h)$ is defined as follows

$$\mathfrak{d}_{\Delta,r}(\ell,h) := \begin{cases} \delta_\ell(h), & \text{if } \Delta = 1, \\ \chi_\Delta((rh)'_\ell), & \text{if } \Delta \neq 1,\ \delta_\ell(rh) = 1 \text{ and } \Delta \mid d(\ell, rh), \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.2}$$

In fact, the constant $\mathfrak{d}_{\Delta,r}(\ell, h)$ always vanishes in the case that N is square-free [AE13, Proposition 5.4].

4. The Bruinier-Funke theta lift

In this chapter we define the Bruinier-Funke theta lift. The construction is similar to the one in the previous chapter but now we employ the Millson theta function as an integration kernel. This lift was first considered by Bruinier and Funke in their fundamental paper [BF04]. The Bruinier-Funke lift behaves like the Kudla-Millson lift in many aspects. In the next chapter we show that the lifts are in some sense dual to each other.

We let $F \in H^{+}_{-2k}(N)$ be a harmonic Maass form of weight $-2k \leq 0$ for $\Gamma_0(N)$. Recall that $\widetilde{\rho} = \rho$ if $\Delta > 0$ and $\widetilde{\rho} = \overline{\rho}$ if $\Delta < 0$. The Bruinier-Funke lift is a weakly holomorphic modular form of weight $3/2 + k$ transforming with respect to $\widetilde{\rho}$ if k is odd. If k is even, it is a harmonic Maass form in $H^{+}_{1/2-k,\widetilde{\rho}}$ and is weakly holomorphic if and only if the twisted L-function of $\xi_{-2k}(F)$ vanishes at $s = k + 1$.

The case $k = 0$ is special. In this case the Millson theta function satisfies a differential equation with respect to the Shintani theta function leading to a relation between the Bruinier-Funke and Shintani theta lift. We make use of this in Chapter 6 to prove results on L-series of weight 2 cusp forms.

We also compute the coefficients of the holomorphic part of the lift which are again given by the twisted traces of CM values of the input function as in Section 2.7.

Especially in the case $k = 0$, it would be interesting to compute the Fourier expansion of the non-holomorphic part as well. However, we can not employ the methods of [BF06] directly which rely on the existence of a Green current for the Kudla-Millson Schwartz function. In our case no such function is known.

As in the previous chapter, we assume the notation of Section 1.2. In particular, V is a rational quadratic space of signature $(1, 2)$ that we identify with the 2×2 matrices in $\mathbb{Q}$ with trace 0. Recall that M is the modular curve $Y_0(N) = \Gamma_0(N) \setminus \mathbb{H}$. We frequently use the identification between the symmetric space D and the complex upper half plane $\mathbb{H}$ as in (1.2.2). As before, z is used as a variable for integer weight forms, and τ is used for half-integer weight forms. Recall that we write $q = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. Let L be the lattice defined in Section 1.2.1, let $\Delta \in \mathbb{Z}$ be a fundamental discriminant, and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. By ρ we denote the Weil representation associated to the lattice L.

4.1. Definition of the Bruinier-Funke theta lift

In this section we define the Bruinier-Funke theta lift. Let $\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}})$ be the Millson theta function defined in Section 2.6.2. It is a non-holomorphic $\mathbb{C}[L'/L]$-valued modular form of weight $1/2$ for the representation $\widetilde{\rho}$ in the variable τ. Furthermore, it is $\Gamma_0(N)$-invariant in the variable $z \in D$. We then make the following definition.

Definition 4.1.1. Let $k \geq 0$ be an integer and let F be a harmonic weak Maass form in $H^+_{-2k}(N)$. For even k we define the Bruinier-Funke lift of F by

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F) = L^{k/2}_{1/2,\tau} \int_M (R^k_{-2k,z}F)(z)\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}})d\mu(z) \tag{4.1.1}$$

and for k odd

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F) = R^{(k+1)/2}_{1/2,\tau} \int_M (R^k_{-2k,z}F)(z)\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}})d\mu(z). \tag{4.1.2}$$

Since the Millson theta function is rapidly decaying at the cusps (Proposition 2.6.5), the integrals above exist.

4.2. Automorphic and analytic properties

Here, we study the Bruinier-Funke lift $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F)$ with respect to its transformation behavior under the Petersson slash operator and the analytic conditions it satisfies on $\mathbb{H}$ and at the cusps. We have the following theorem.

Theorem 4.2.1. *Let $k \geq 0$ be an integer and let $F \in H^+_{-2k}(N)$ be a harmonic Maass form of weight $-2k$ for $\Gamma_0(N)$. The Bruinier-Funke theta lift of F has the following properties:*

(i) If $k = 0$, the Bruinier-Funke lift of F is a harmonic Maass form of weight $1/2$ transforming with respect to the representation $\widetilde{\rho}$.

(ii) For square-free N and odd $k > 0$ the Bruinier-Funke theta lift of F is a weakly holomorphic form of weight $3/2 + k$ transforming with respect to the representation $\widetilde{\rho}$.

(iii) For square-free N and even $k > 0$ the Bruinier-Funke theta lift of F is a harmonic Maass form of weight $1/2 - k$ transforming with respect to the representation $\widetilde{\rho}$. Moreover, the lift $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau, F)$ is a weakly holomorphic modular form if and only if the twisted L-function of $\xi_{-2k}(F) \in S_{3/2+k}(N)$ vanishes at $s = k + 1$.

Note that the transformation properties of the twisted theta function $\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}})$ directly imply that the lift transforms with representation $\widetilde{\rho}$. First we investigate how the lift behaves under the action of the Laplace operator. As in the case of the Kudla-Millson theta lift, we then show that the lift satisfies the correct growth conditions at the cusps if N is square-free by computing the lift of Poincaré series and using the equivariance under the action of the Atkin-Lehner involutions.

The case $F \in H_0^+(N)$ is special. Then the Bruinier-Funke lift is related to the Shintani theta lift via the differential equation between the two theta functions (see Equation (2.6.12)). This directly implies that the lift is a harmonic Maass form (for all N) if the constant coefficients of F at all cusps vanish. In this case, we compute the lift of the constant function to show that the resulting function is contained in $H^+_{1/2,\widetilde{\rho}}$.

Lemma 4.2.2. *Let F be an eigenform of $\Delta_{-2k,z}$ with eigenvalue λ. Then the Bruinier-Funke lift of F is an eigenform of eigenvalue $\frac{\lambda}{4}$ of $\Delta_{1/2-k,\tau}$ if k is even and of $\Delta_{3/2+k,\tau}$ if k is odd.*

Proof. For even k we have by Lemma 2.3.9

$$\begin{aligned}&\Delta_{1/2-k,\tau}\left(L^{k/2}_{1/2,\tau}\int_M (R^k_{-2k,z}F)(z)\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}})d\mu(z)\right)\\&= L^{k/2}_{1/2,\tau}\int_M (R^k_{-2k,z}F)(z)\Delta_{1/2,\tau}\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}})d\mu(z) + \frac{k}{4}(k+1)\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F).\end{aligned}$$

The relation of the action of $\Delta_{0,z}$ and $\Delta_{1/2,\tau}$ on $\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}})$ (see Equation (2.6.11)) implies that

$$\Delta_{1/2-k,\tau}\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F) = \frac{1}{4}L^{k/2}_{1/2,\tau}\int_M (R^k_{-2k,z}F)(z)\Delta_{0,z}\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}})d\mu(z) + \frac{k}{4}(k+1)\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F).$$

By the square exponential decay of the Millson theta function (Proposition 2.6.5) we may move the Laplacian to $(R^k_{-2k,z}F)(z)$. Using that $\Delta_{0,z}(R^k_{-2k,z}F) = R^k_{-2k,z}(\Delta_{-2k,z} - k(k+1))$ (Lemma 2.3.9) and that F has eigenvalue λ under $\Delta_{-2k,z}$ we then obtain the result. For odd k we proceed analogously. □

As for the Kudla-Millson lift, we compute the lift of Poincaré series. Recall that $\epsilon = 1$ if $\Delta > 0$ and $\epsilon = i$ if $\Delta < 0$.

Theorem 4.2.3. *For even k we have*

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F_m(z,s,-2k)) = C_e \sum_{n|m}\left(\frac{\Delta}{n}\right)n^k\mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|,-\frac{m}{n}r}\left(\tau,\frac{s}{2}+\frac{1}{4},\frac{1}{2}-k\right),$$

where

$$C_e = -\frac{2^{2k-s+1}\bar{\epsilon}}{\Gamma(s/2)i}\pi^{k/2+1/2}N^{k/2+1/2}\left|\Delta\right|^{1/2-k/2}\prod_{j=0}^{k-1}(s+j-k)\prod_{j=0}^{k/2-1}\left(\frac{s-1}{2}-j\right),$$

and for odd k we have

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F_m(z,s,-2k)) = C_o \sum_{n|m}\left(\frac{\Delta}{n}\right)n^{-(k+1)}\mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|,-\frac{m}{n}r}\left(\tau,\frac{s}{2}+\frac{1}{4},\frac{1}{2}-k\right),$$

where

$$C_o = -\frac{2^{2s+2k-1}\Gamma(s/2+1/2)\,\bar{\epsilon}}{\Gamma(2s)i}\pi^{3k/2}N^{-k/2}\left|\Delta\right|^{k/2+1}m^{2k+1}\prod_{j=0}^{k-1}(s+j-k)\prod_{j=0}^{(k-1)/2}\left(\frac{s+1}{2}+j\right).$$

Proof. We first compute the lift of the weight 0 Poincaré series $F_m(z,s,0)$ and prove that

$$\mathcal{I}_{\Delta,r}(\tau,F_m(z,s,0)) = \frac{2^{-s+1}i}{\Gamma(s/2)}\sqrt{\pi N\left|\Delta\right|\bar{\epsilon}}\sum_{n|m}\left(\frac{\Delta}{n}\right)\mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|,-\frac{m}{n}r}\left(\tau,\frac{s}{2}+\frac{1}{4},\frac{1}{2}\right). \qquad (4.2.1)$$

Using the definition of the Poincaré series (2.5.1) and an unfolding argument we obtain

$$\mathcal{I}_{\Delta,r}(\tau,F_m(z,s,0)) = \frac{1}{\Gamma(2s)}\int_{\Gamma_\infty\backslash\mathbb{H}}\mathcal{M}_{s,0}(4\pi my)e(-mx)\Theta_{\Delta,r}(\tau,z,\psi_{\mathrm{KM}})d\mu(z).$$

By Proposition 2.6.7 this equals

$$-\frac{\bar{\epsilon}\,N}{\Gamma(2s)2i}\sum_{n=1}^{\infty}\left(\frac{\Delta}{n}\right)n\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}I(\tau,s,m,n)|_{1/2,\widetilde{\rho}_K}\ \gamma,$$

where

$$I(\tau,s,m,n) = \int_{y=0}^{\infty}\int_{x=0}^{1}y^2\mathcal{M}_{s,0}(4\pi my)e(-mx)\exp\left(-\frac{\pi n^2Ny^2}{|\Delta|\,v}\right)$$
$$\times\, v^{-1/2}\sum_{\lambda\in K'}e\left(\left|\Delta\right|Q(\lambda)\bar{\tau}-2N\lambda nx\right)\mathfrak{e}_{r\lambda}\frac{dxdy}{y^2}.$$

Identifying $K' = \mathbb{Z}\begin{pmatrix}1/2N & 0\\ 0 & -1/2N\end{pmatrix}$ we find that

$$\sum_{\lambda\in K'}e\left(\left|\Delta\right|Q(\lambda)\bar{\tau}-2N\lambda nx\right)\mathfrak{e}_{r\lambda} = \sum_{b\in\mathbb{Z}}e\left(-\left|\Delta\right|\frac{b^2}{4N}\bar{\tau}-nbx\right)\mathfrak{e}_{rb}.$$

Inserting this in the formula for $I(\tau,s,m,n)$, and integrating over x, we see that $I(\tau,s,m,n)$ vanishes whenever $n\nmid m$ and the only summand occurs for $b=-m/n$ when $n\mid m$. Thus, $I(\tau,s,m,n)$ equals

$$v^{-1/2}e\left(-\left|\Delta\right|\frac{m^2}{4Nn^2}\bar{\tau}\right)\ \cdot\ \int_{y=0}^{\infty}\mathcal{M}_{s,0}(4\pi my)\exp\left(-\frac{\pi n^2Ny^2}{|\Delta|\,v}\right)dy\ \mathfrak{e}_{-rm/n}. \qquad (4.2.2)$$

As in the proof of Theorem 3.2.4 we use that

$$\mathcal{M}_{s,0}(4\pi my) = 2^{2s-1}\Gamma\left(s+\frac{1}{2}\right)\sqrt{4\pi my}\cdot I_{s-1/2}(2\pi my).$$

Substituting $t = y^2$ yields

$$\begin{aligned}
&\int_{y=0}^{\infty} \mathcal{M}_{s,0}(4\pi m y) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta|\, v}\right) dy \\
&= 2^{2s-1}\Gamma\left(s+\frac{1}{2}\right) \int_{y=0}^{\infty} \sqrt{4\pi m y}\; I_{s-1/2}(2\pi m y) \exp\left(-\frac{\pi n^2 N y^2}{|\Delta|\, v}\right) dy \\
&= 2^{2s-1}\Gamma\left(s+\frac{1}{2}\right) \sqrt{m\pi} \int_{t=0}^{\infty} t^{-1/4} I_{s-1/2}(2\pi m t^{1/2}) \exp\left(-\frac{\pi n^2 N t}{|\Delta|\, v}\right) dt.
\end{aligned}$$

Again, the last integral is a Laplace transform and is computed in [EMOT54] (see (20) on p. 197). It equals

$$\frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(s+\frac{1}{2}\right)} (\pi m)^{-1} \left(\frac{\pi n^2 N}{|\Delta|\, v}\right)^{-1/4} \exp\left(\frac{\pi m^2 |\Delta|\, v}{2n^2 N}\right) M_{-\frac{1}{4},\frac{s}{2}-\frac{1}{4}} \left(\frac{\pi m^2 |\Delta|\, v}{n^2 N}\right).$$

Therefore, we have that $I(\tau, s, m, n)$ equals

$$2^{2s-1}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) \sqrt{\frac{|\Delta|}{\pi N n^2}} e\left(-\frac{m^2 |\Delta|\, u}{4n^2 N}\right) \mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{1}{2}} \left(\frac{\pi m^2 |\Delta|\, v}{n^2 N}\right) \mathfrak{e}_{-rm/n}.$$

Putting everything together we obtain the following for the lift of $F_m(z, s, 0)$

$$\begin{aligned}
&-\frac{2^{2s-2}\Gamma(s/2+1/2)\bar{\epsilon}}{\Gamma(2s) i} \sqrt{\frac{N\,|\Delta|}{\pi}} \sum_{n|m} \left(\frac{\Delta}{n}\right) \\
&\times \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \left[e\left(-\frac{m^2 |\Delta|\, u}{4Nn^2}\right) \mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{1}{2}} \left(\frac{\pi m^2 |\Delta|\, v}{n^2 N}\right) \mathfrak{e}_{-rm/n} \right] \Bigg|_{1/2,\widetilde{\rho}_K} \gamma,
\end{aligned}$$

which implies the formula in (4.2.1).

A repeated application of Proposition 2.5.1 yields that

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F_m(z,s,-2k)) = (4\pi m)^k \prod_{j=0}^{k-1} (s+j-k) \int_M F_m(z,s,0) \Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}}) d\mu(z).$$

For even k we have by Proposition 2.5.2

$$\begin{aligned}
&L^{k/2}_{1/2,\tau} \left(\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{1}{2}} \left(\frac{\pi m^2 |\Delta|\, v}{n^2 N}\right) e\left(-\frac{m^2 |\Delta|\, u}{4Nn^2}\right)\right) \\
&= \left(\frac{Nn^2}{\pi m^2 |\Delta|}\right)^{k/2} \prod_{j=0}^{k/2-1} \left(\frac{s-1}{2} - j\right) \mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{1}{2}-k} \left(\frac{\pi m^2 |\Delta|\, v}{n^2 N}\right) e\left(-\frac{m^2 |\Delta|\, u}{4Nn^2}\right).
\end{aligned}$$

Since the lowering and the slash operator commute, we obtain

$$\mathcal{I}_{\Delta,r}(\tau, F_m(z,s,-2k)) = C_{\mathrm{e}} \cdot \sum_{n|m} \left(\frac{\Delta}{n}\right) n^k \mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|,-\frac{m}{n}r}\left(\tau, \frac{s}{2}+\frac{1}{4}, \frac{1}{2}-k\right),$$

with

$$C_{\mathrm{e}} = -\frac{2^{2k-s+1}\bar{\epsilon}}{\Gamma(s/2)i}\pi^{k/2+1/2}N^{k/2+1/2}|\Delta|^{1/2-k/2}\prod_{j=0}^{k-1}(s+j-k)\prod_{j=0}^{k/2-1}\left(\frac{s-1}{2}-j\right).$$

For odd k we compute

$$\begin{aligned}
&R_{1/2,\tau}^{(k+1)/2}\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{1}{2}}\left(\frac{\pi m^2|\Delta|v}{n^2N}\right)e\left(-\frac{m^2|\Delta|u}{4Nn^2}\right)\\
&= \left(\frac{\pi m^2|\Delta|}{Nn^2}\right)^{(k+1)/2}\prod_{j=0}^{(k-1)/2}\left(\frac{s+1}{2}+j\right)\mathcal{M}_{\frac{s}{2}+\frac{1}{4},\frac{3}{2}+k}\left(\frac{\pi m^2|\Delta|v}{n^2N}\right)e\left(-\frac{m^2|\Delta|u}{4Nn^2}\right).
\end{aligned}$$

and get

$$\mathcal{I}_{\Delta,r}(\tau, F_m(z,s,-2k)) = C_{\mathrm{o}} \cdot \sum_{n|m} \left(\frac{\Delta}{n}\right) n^{-(k+1)} \mathcal{F}_{\frac{m^2}{4Nn^2}|\Delta|,-\frac{m}{n}r}\left(\tau, \frac{s}{2}+\frac{1}{4}, \frac{1}{2}-k\right),$$

with

$$C_{\mathrm{o}} = -\frac{2^{2s+2k-1}\bar{\epsilon}\Gamma(s/2+1/2)}{\Gamma(2s)i}\pi^{3k/2}N^{-k/2}|\Delta|^{k/2+1}m^{2k+1}\prod_{j=0}^{k-1}(s+j-k)\prod_{j=0}^{(k-1)/2}\left(\frac{s+1}{2}+j\right).$$

□

From Proposition 2.6.6 we can directly deduce the following Proposition on the action of the Atkin-Lehner involutions on the Bruinier-Funke theta lift.

Proposition 4.2.4. *For an Atkin-Lehner involution W_Q^N as in Definition 1.2.2, $h \in L'/L$ and $F \in H_{-2k}(N)$, we have*

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r,W_Q^N.h}(\tau, F) = \mathcal{I}^{\mathrm{BF}}_{\Delta,r,h}(\tau, F|_{-2k}(W_Q^N)^{-1}).$$

The following two results follow completely analogous to the corresponding results in the previous chapter (Corollary 3.2.6 and Theorem 3.2.7).

Corollary 4.2.5. *Let N be square-free, $k > 0$ an integer and $F \in H^+_{-2k}(N)$. Then the Bruinier-Funke lift of F is a weakly holomorphic modular form of weight $3/2+k$ if k is odd. If k is even, then the lift is a harmonic Maass form of weight $1/2-k$.*

Theorem 4.2.6. *Let N be square-free and $k \neq 0$ be even. For a harmonic Maass form $F \in H^+_{-2k}(N)$ the lift is weakly holomorphic if and only if*

$$L(\xi_{-2k}(F), \Delta, k+1) = 0.$$

In particular, this is the case when F is weakly holomorphic.

Proof of part (ii) and (iii) of Theorem 4.2.1. Combining the results proved above we obtain the second and third part of Theorem 4.2.1. □

4.2.1. Relation to the Shintani lift

The Bruinier-Funke lift of a harmonic Maass form of weight 0 and the Shintani lift of $\xi_0(F)$ are closely related as we will explain now.

We let $\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau, G)$ be the Shintani lifting of a cusp form G of weight 2 for $\Gamma_0(N)$. It is defined as

$$\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau, G) = \int_M G(z)\overline{\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})}y^2 d\mu(z)$$

and is a vector valued modular form of weight 3/2 transforming with respect to the representation $\overline{\widetilde{\rho}}$.

We then have the following relation between the two theta lifts.

Theorem 4.2.7. *Let $F \in H^+_0(N)$ with vanishing constant term at all cusps of $\Gamma_0(N)$. Then we have that*

$$\xi_{1/2,\tau}\left(\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F)\right) = -\frac{1}{2\sqrt{N}}\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau, \xi_{0,z}(F)).$$

Proof. By Stokes' theorem and Lemma 2.3.11 we have that

$$\begin{aligned}\mathcal{I}^{\mathrm{Sh}}_{\Delta,r}(\tau, \xi_{0,z}(F)) &= \int_M \xi_0(F(z))\overline{\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})}y^2 d\mu(z)\\ &= -\int_M \overline{F(z)}\xi_{2,z}(\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}}))d\mu(z) + \lim_{t\to\infty}\int_{\partial\mathcal{F}_t} \overline{F(z)\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})}d\bar{z},\end{aligned}$$

where $\mathcal{F}_t = \{z \in \mathbb{H} : \Im(z) \leq t\}$ denotes the truncated fundamental domain. The differential equation between the Shintani and the Millson theta function (Equation (2.6.12)) implies that

$$\begin{aligned}&-\int_M \overline{F(z)}\xi_{2,z}(\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}}))d\mu(z)\\ &= -\frac{1}{2\sqrt{N}}\int_M \overline{F(z)}\xi_{1/2,\tau}(\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}}))d\mu(z) = -\frac{1}{2\sqrt{N}}\xi_{1/2,\tau}\left(\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F)\right).\end{aligned}$$

It remains to show that

$$\lim_{t\to\infty}\int_{\partial\mathcal{F}_t} \overline{F(z)\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})}d\bar{z} = 0.$$

We have to investigate the growth of the Shintani theta function at the cusps. Again, we let $\Delta = N = 1$, $L = \mathbb{Z}^3$, and $h' = 0, 1/2$ for simplicity and obtain

$$\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}}) = \sum_{\substack{a,c\in\mathbb{Z}\\ b\in\mathbb{Z}+h'}} -\frac{c\bar{z}^2 - b\bar{z} + a}{4y^2} e^{-\frac{\pi v}{y^2}(c|z|^2 - bx + a)} e^{2\pi i\bar{\tau}(-b^2/4+ac)}.$$

As in the proof of Proposition 2.6.5, we apply Poisson summation to the sum over a. Thus, we consider

$$\int_{-\infty}^{\infty} -\frac{c\bar{z}^2 - b\bar{z} + a}{4y^2} e^{-\frac{\pi v}{y^2}(c|z|^2 - bx + a)} e^{2\pi i\bar{\tau}(-b^2/4+ac)} e^{2\pi i w a} da.$$

Proceeding as in the proof of Proposition 2.6.5, we obtain

$$\theta_h(\tau, z, \varphi_{\mathrm{Sh}}) = -\frac{1}{4\sqrt{v}y} \sum_{\substack{w,c\in\mathbb{Z}\\ b\in\mathbb{Z}+h'}} e^{-2\pi i\bar{\tau}(b/2-cx)^2} e^{2\pi i(bxw - cx^2 w)}$$
$$\times \left(c\bar{z}^2 + biy - c|z|^2 + i\frac{y^2}{v}(c\bar{\tau} + w)\right) e^{-\frac{\pi y^2}{v}|c\tau+w|^2}.$$

If c and w are not both equal to 0 this vanishes in the limit as $y \to \infty$. In this case, the whole integral vanishes. But if $c = w = 0$ we have

$$-\frac{i}{4\sqrt{v}} \sum_{b\in\mathbb{Z}+h'} b e^{\pi i\bar{\tau}b^2/2}.$$

Thus, we are left with (the complex conjugate of)

$$\int_{\partial\mathcal{F}_t} F(z)\Theta(\tau, z, \varphi_{\mathrm{Sh}}) dz = \frac{i}{4\sqrt{v}} \sum_{b\in\mathbb{Z}+h'} b e^{\pi i\bar{\tau}b^2/2} \int_0^1 F(x+it)dx.$$

We see that

$$\lim_{t\to\infty} \int_0^1 F(x+it)dx = 0,$$

since the constant coefficient of F at the cusp ∞ vanishes. Therefore,

$$\lim_{t\to\infty} \int_{\partial\mathcal{F}_t} \overline{F(z)\Theta(\tau, z, \varphi_{\mathrm{Sh}})} d\bar{z} = 0.$$

Generalizing to arbitrary N, similar growth estimates hold for the other cusps of M. □

The relation to the Shintani lifting directly implies

Proposition 4.2.8. *Let $F \in H_0^+(N)$ with vanishing constant term at all cusps and let $\xi_{0,z}(F) = G \in S_2^{\mathrm{new}}(N)$. The lift $\mathcal{I}_{\Delta,r}^{\mathrm{BF}}(\tau, F)$ is weakly holomorphic if and only if the*

Shintani lift of G vanishes, i.e. if

$$L(G, \Delta, 1) = 0.$$

In particular, this happens if F is weakly holomorphic.

4.2.2. The lift of the constant function

We now turn to the computation of the lift of the constant function. We follow the strategy of Bruinier and Funke [BF06] and first compute the lift of the (normalized) real-analytic Eisenstein series $\mathcal{E}_0(z,s)$ of weight 0 for $\Gamma_0(N)$ and then take residues at $s = \frac{1}{2}$. The Eisenstein series $\mathcal{E}_0(z,s)$ is given by

$$\mathcal{E}_0(z,s) = \frac{1}{2}\zeta^*(2s+1) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\Im(\gamma z))^{s+\frac{1}{2}},$$

where

$$\zeta^*(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

denotes the completed Riemann Zeta function. Here, $\Gamma_\infty = \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. The Eisenstein series $\mathcal{E}_0(z,s)$ converges for $\Re(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = \frac{1}{2}$ with residue

$$\frac{\pi}{6\mathrm{Vol}(\Gamma_0(N) \setminus \mathbb{H})} = \frac{1}{2N \prod_{p|N} \left(1 + \frac{1}{p}\right)}.$$

For the computation of the lift of a constant we need the following results on the Eisenstein series of weight 1/2 defined by

$$\mathcal{E}_{1/2,K}(\tau, s) = \frac{1}{2} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} (v^{\frac{1}{2}(s+\frac{1}{2})}\mathfrak{e}_0)|_{1/2,\widetilde{\rho}_K}\gamma.$$

Here, K is the sublattice $\mathbb{Z}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Note that $\mathcal{E}_{1/2,K}(\tau,s)$ vanishes if $\widetilde{\rho}_K = \rho_K$, i.e. if $\Delta > 0$. This can bee seen by replacing γ by $Z\gamma$ in the sum, where $Z = (-1, i) \in \mathrm{Mp}_2(\mathbb{Z})$ and using that $\mathfrak{e}_0|_{1/2,\rho_K} Z = -\mathfrak{e}_0$. Combining Theorem 6.2, the results of Section 6.4 and Equation (5.19) in [BFI13] we obtain the following proposition.

Proposition 4.2.9. *The residue of $\mathcal{E}_{1/2,\widetilde{\rho}_K}(\tau, s)$ at $s = \frac{1}{2}$ is given by*

$$\frac{12}{\pi} \sum_{\ell \in \Gamma_0(N) \backslash \mathrm{Iso}(V)} \frac{B_\ell(1)}{\sqrt{N}} \tilde{\Theta}_{K_\ell}(\tau).$$

Here, $B_\ell(s)$ is a holomorphic function associated to the Fourier expansion at the cusp ℓ of a weight 0 Eisenstein series (see [BFI13, Section 5.5.]) and $\tilde{\Theta}_{K_\ell}(\tau)$ is a theta series *associated to the cusp ℓ (see [BFI13, Section 2]).*

Theorem 4.2.10. *Let $\Delta < 0$. For the lift of the weight 0 Eisenstein series we obtain*

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \mathcal{E}_0(z,s)) = -\zeta^*(2s+1)N^{1/4-s/2}|\Delta|^{s/2+3/4}\frac{\bar{\epsilon}}{2i\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)}$$
$$\times \Lambda\left(\epsilon_\Delta, s+\frac{1}{2}\right)\mathcal{E}_{1/2,\bar{\rho}_K}(\tau, s).$$

Remark 4.2.11. Note that the vanishing of the Millson theta function for some values of N and Δ implies the vanishing of the Bruinier-Funke lift as well. For example, if $N = \Delta = 1$, the lift vanishes. The same holds for $N = 1$ and $\Delta > 0$.

Proof. The proof follows the one in [BF06, Theorem 7.1, Corollary 7.2] and [AE13, Theorem 6.1]. Using the standard unfolding trick we obtain

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \mathcal{E}_0(z,s)) = \zeta^*(2s+1)\int_{\Gamma_\infty\backslash\mathbb{H}}\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}})y^{s+\frac{1}{2}}d\mu(z).$$

By Proposition 2.6.7 this equals

$$-\zeta^*(2s+1)\frac{N\bar{\epsilon}}{2i}\sum_{n\geq 1}n\left(\frac{\Delta}{n}\right)\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}\frac{1}{(c\tau+d)^{\frac{1}{2}}}\widetilde{\rho}_K^{-1}(\gamma)\frac{1}{\Im(\gamma\tau)^{1/2}}$$
$$\times\int_{y=0}^{\infty}y^{s+\frac{1}{2}}\exp\left(-\frac{N\pi n^2y^2}{|\Delta|\,\Im(\gamma\tau)}\right)dy$$
$$\times\int_{x=0}^{1}\sum_{\lambda\in K'}e\left(|\Delta|\,Q(\lambda)\bar{\tau}-2N\lambda nx\right)\mathfrak{e}_{r\lambda}dx.$$

The integral over x equals $\mathfrak{e}_0$ and the one over y equals

$$\frac{1}{2}\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\left(|\Delta|\,\Im(\gamma\tau)\right)^{\frac{s}{2}+\frac{3}{4}}(N\pi)^{-\frac{s}{2}-\frac{3}{4}}n^{-s-\frac{3}{2}}.$$

Thus, we have

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \mathcal{E}_0(z,s)) = -\zeta^*(2s+1)N^{-\frac{s}{2}+\frac{1}{4}}\bar{\epsilon}\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\frac{1}{2i}|\Delta|^{\frac{s}{2}+\frac{3}{4}}\pi^{-\frac{s}{2}-\frac{3}{4}}$$
$$\times L\left(\epsilon_\Delta, s+\frac{1}{2}\right)\left(\frac{1}{2}\sum_{\gamma\in\widetilde{\Gamma}_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}\left(v^{\frac{1}{2}(s+\frac{1}{2})}\mathfrak{e}_0\right)|_{1/2,K}\gamma\right)$$

For $\Delta > 0$ this vanished and for $\Delta < 0$ we find that this equals

$$-\zeta^*(2s+1)\Lambda\left(\epsilon_\Delta, s+\frac{1}{2}\right)\mathcal{E}_{1/2,K}(\tau,s)N^{1/4-s/2}|\Delta|^{s/2+3/4}\frac{\bar{\epsilon}}{2i\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)}.$$

□

We now take residues at $s = \frac{1}{2}$ in Theorem 4.2.10 to compute the lift of the constant function.

Theorem 4.2.12. *For $\Delta < 0$ we have*

$$\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,1) = -\frac{2\,N\prod_{p|N}\left(1+\frac{1}{p}\right)\bar{\epsilon}}{\pi i}\,|\Delta|\;\Lambda(\epsilon_\Delta,1)\sum_{\ell\in\Gamma_0(N)\backslash\mathrm{Iso}(V)}\frac{B_\ell(1)}{\sqrt{N}}\tilde{\Theta}_{K_\ell}(\tau).$$

Remark 4.2.13. Note that $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,1) \in M_{1/2,\widetilde{\rho}_L}$.

Remark 4.2.14. This might be interpreted as a first term identity in the sense of Kudla and Rallis.

Proof of part (i) of Theorem 4.2.1. Combining the preceding results we obtain the statements in the first part of Theorem 4.2.1. □

4.3. Fourier coefficients of the holomorphic part

Now we turn to the computation of the Fourier coefficients of positive index of the holomorphic part of the theta lift. Recall that the modular trace functions $\mathbf{t}^+_{\Delta,r}(F;m,h)$ and $\mathbf{t}^-_{\Delta,r}(F;m,h)$ are defined as

$$\mathbf{t}^+_{\Delta,r}(F;m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L^+_{|\Delta|m,rh}}\frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|}\partial F(D_\lambda)$$

$$\mathbf{t}^-_{\Delta,r}(F;m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L^-_{|\Delta|m,rh}}\frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|}\partial F(D_\lambda),$$

where $\partial F := R^k_{-2k}(F)$ for $F \in H^+_{-2k}(N)$ with $k \geq 0$, $h \in L'/L$ and $m \in \mathbb{Q}_{>0}$ with $m \equiv \mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})$ (see Section 2.7).

Theorem 4.3.1. *We let $k \geq 0$ be an integer and $F \in H^+_{-2k}(N)$. Moreover, let $h \in L'/L$ and $m \in \mathbb{Q}_{>0}$ with $m \equiv \mathrm{sgn}(\Delta)Q(h)\,(\mathbb{Z})$. We obtain the following results:*

(i) Let $k \geq 0$ be even. Then the coefficient of index (m,h) of the holomorphic part of the lift $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)$ is given by

$$\frac{\sqrt{|\Delta|}}{2\sqrt{m}}\left(\frac{|\Delta|}{4\pi m}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{k+1}{2}+j\right)\left(j-\frac{k}{2}\right)\left(\mathbf{t}^{+}_{\Delta,r}(F;m,h)-\mathbf{t}^{-}_{\Delta,r}(F;m,h)\right).$$

(ii) Let k be odd. Then the coefficient of index (m,h) of the holomorphic part of the lift $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)$ is given by

$$\frac{\sqrt{|\Delta|}}{2\sqrt{m}}\left(-\frac{4\pi m}{|\Delta|}\right)^{(k+1)/2}\left(\mathbf{t}^{+}_{\Delta,r}(F;m,h)-\mathbf{t}^{-}_{\Delta,r}(F;m,h)\right).$$

Proof. The proof is very similar to the one of Theorem 3.3.1 and we frequently omit some arguments that are completely analogous to the ones in the proof of Theorem 3.3.1. Again, we first prove the result for $\Delta = 1$. Using the methods developed in [AE13] we then deduce the general result. We write

$$\begin{aligned}\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F) &= \sum_{h\in L'/L}\sum_{m\in\mathbb{Q}}\left(\sum_{\lambda\in L_{m,h}}\int_M \partial F(z)\sqrt{v}\psi^0_{\mathrm{KM}}(\sqrt{v}\lambda,z)d\mu(z)\right)e^{2\pi i m\tau}\\ &= \sum_{h\in L'/L}\sum_{m\in\mathbb{Q}} C(m,h)e^{2\pi i m\tau}\end{aligned}$$

for the Fourier expansion of $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)$. By the usual unfolding argument we obtain

$$\begin{aligned}C(m,h) = &\sum_{\lambda\in\Gamma_0(N)\backslash L^+_{m,h}}\frac{1}{|\overline{\Gamma}_\lambda|}\int_M \partial F(z)\sqrt{v}\psi^0_{\mathrm{KM}}(\sqrt{v}\lambda,z)d\mu(z)\\ &+\sum_{\lambda\in\Gamma_0(N)\backslash L^-_{m,h}}\frac{1}{|\overline{\Gamma}_\lambda|}\int_M \partial F(z)\sqrt{v}\psi^0_{\mathrm{KM}}(\sqrt{v}\lambda,z)d\mu(z).\end{aligned}$$

For the latter sum we find that it equals

$$-\sum_{\lambda\in\Gamma_0(N)\backslash L^-_{m,h}}\frac{1}{|\overline{\Gamma}_{-\lambda}|}\int_M \partial F(z)\sqrt{v}\psi^0_{\mathrm{KM}}(-\sqrt{v}\lambda,z)d\mu(z).$$

Note that we have to distinguish between elements in $L^+_{m,h}$ and $L^-_{m,h}$ here since ψ_{KM} is an odd function in the first variable. Proceeding as in the proof of Theorem 3.3.1 we obtain

$$\begin{aligned} C(m,h) = &\sum_{\lambda\in\Gamma_0(N)\backslash L^+_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|}\int_{G(\mathbb{R})} \partial F(gi)\sqrt{v}\psi^0_{\mathrm{KM}}(\sqrt{v}\lambda, gi)dg \\ &- \sum_{\lambda\in\Gamma_0(N)\backslash L^-_{m,h}} \frac{1}{|\overline{\Gamma}_{-\lambda}|}\int_{G(\mathbb{R})} \partial F(gi)\sqrt{v}\psi^0_{\mathrm{KM}}(-\sqrt{v}\lambda, gi)dg. \end{aligned}$$

Since the group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $L^+_{m,h}$, there is a $g_1 \in \mathrm{SL}_2(\mathbb{R})$ such that $g_1^{-1}.\lambda = \sqrt{m}\lambda(i)$ for $\lambda \in L^+_{m,h}$. Also, there is a $g_1 \in \mathrm{SL}_2(\mathbb{R})$ such that $g_1^{-1}.(-\lambda) = \sqrt{m}\lambda(i)$ for $\lambda \in L^-_{m,h}$. We then have

$$\begin{aligned} C(m,h) = &\sum_{\lambda\in\Gamma_0(N)\backslash L^+_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|}\int_{G(\mathbb{R})} \partial F(g_1gi)\sqrt{v}\psi^0_{\mathrm{KM}}\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right)dg \\ &- \sum_{\lambda\in\Gamma_0(N)\backslash L^-_{m,h}} \frac{1}{|\overline{\Gamma}_{-\lambda}|}\int_{G(\mathbb{R})} \partial F(g_1gi)\sqrt{v}\psi^0_{\mathrm{KM}}\left(\sqrt{v}\sqrt{m}g^{-1}.\lambda(i), i\right)dg. \end{aligned}$$

Using the Cartan decomposition of $\mathrm{SL}_2(\mathbb{R})$ we find

$$C(m,h) = \sum_{\lambda\in\Gamma_0(N)\backslash L^+_{m,h}} \frac{1}{|\overline{\Gamma}_\lambda|}\partial F(D_\lambda)\sqrt{v}Y_c(\sqrt{mv}) - \sum_{\lambda\in\Gamma_0(N)\backslash L^-_{m,h}} \frac{1}{|\overline{\Gamma}_{-\lambda}|}\partial F(D_{-\lambda})\sqrt{v}Y_c(\sqrt{mv}),$$

with

$$Y_c(t) = 4\pi\int_1^\infty \psi^0_{\mathrm{KM}}(t\alpha(a)^{-1}.\lambda(i), i)\omega_c(\alpha(a))\frac{a^2-a^{-2}}{2}\frac{da}{a}.$$

As before, $\omega_c(\alpha(a)) = \omega_c\left(\frac{a^2+a^{-2}}{2}\right)$ is the spherical function of eigenvalue $c = -k(k+1)$ given by the Legendre polynomial $P_k(x)$. Substituting $a = e^{r/2}$ we obtain

$$Y_c(t) = 4\pi t\int_0^\infty \cosh(r)\sinh(r)P_k(\cosh(r))e^{-4\pi t^2\sinh(r)^2}dr.$$

Setting $x = \sinh(r)^2$ we get

$$Y_c(t) = 2\pi t\int_0^\infty P_k(\sqrt{1+x})e^{-4\pi t^2 x}dx.$$

This is a Laplace transformation computed in equation (7) on page 180 in [EMOT54]. It equals

$$\frac{2\pi t}{(4\pi t^2)^{5/4}}W_{\frac{1}{4},\frac{k}{2}+\frac{1}{4}}(4\pi t^2)e^{2\pi t^2}.$$

We have

$$\sqrt{v}Y_c(\sqrt{mv})e^{2\pi i m\tau} = \frac{1}{2\sqrt{m}}\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}}(4\pi mv)e(mx)$$

and

$$\begin{aligned}
&L_{1/2}^{k/2}\left(\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}}(4\pi mv)e(mx)\right)\\
&= \left(\frac{1}{4\pi m}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{k+1}{2}+j\right)\left(j-\frac{k}{2}\right)\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}-k}(4\pi mv)e(mx)\\
&= \left(\frac{1}{4\pi m}\right)^{k/2}\prod_{j=0}^{k/2-1}\left(\frac{k+1}{2}+j\right)\left(j-\frac{k}{2}\right)e^{2\pi i m\tau}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
R_{1/2}^{(k+1)/2}\left(\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}}(4\pi mv)e(mx)\right) &= (-4\pi m)^{(k+1)/2}\,\frac{1}{\sqrt{m}}\mathcal{W}_{\frac{k}{2}+\frac{3}{4},\frac{1}{2}-k}(4\pi mv)e(mx)\\
&= (-4\pi m)^{(k+1)/2}\,e^{2\pi i m\tau}.
\end{aligned}$$

We now twist this result as described in the proof of Theorem 3.3.1 to obtain the results stated in the theorem. □

5. Duality results for the Kudla-Millson and Bruinier-Funke lift

In this chapter we consider Kudla-Millson and Bruinier-Funke theta lifts that are weakly holomorphic of weight $3/2+k$. We show that the lifts are orthogonal to cusp forms with respect to the Petersson inner product in this case.

Recall that $\widetilde{\rho}=\rho$ if $\Delta>0$ and $\widetilde{\rho}=\overline{\rho}$ if $\Delta<0$. We obtain that the bilinear pairing defined in Section 2.3.2 of the Kudla-Millson or Bruinier-Funke lift $\mathcal{I}(\tau,F)\in M^!_{3/2+k,\widetilde{\rho}}$ with a harmonic Maass form f in the dual space $H^+_{1/2-k,\overline{\widetilde{\rho}}}$ vanishes, i.e.

$$\{\mathcal{I}(\tau,F),f\}=(\mathcal{I}(\tau,F),\xi_{1/2-k}(f))^{\mathrm{reg}}_{3/2+k,\widetilde{\rho}}=0.$$

Recall that $\xi_{1/2-k}(f)$ is a cusp form of weight $3/2+k$ transforming with respect to $\widetilde{\rho}$.

Together with the formula for the bilinear pairing given in Proposition 2.3.20 we obtain formulas for the coefficients of the holomorphic part of f in terms of the coefficients of the holomorphic part of $\mathcal{I}(\tau,F)$, thus in terms of twisted traces of CM values of F.

Choosing $f=\mathcal{I}^{\mathrm{KM}}(\tau,F)\in H^+_{1/2-k,\overline{\widetilde{\rho}}}$ and $\mathcal{I}^{\mathrm{BF}}(\tau,F)\in M^!_{3/2+k,\widetilde{\rho}}$ (or vice versa) we obtain duality results in the spirit of [Zag02].

Recall that M is the modular curve $Y_0(N)=\Gamma_0(N)\setminus\mathbb{H}$. As before, z is used as a variable for integer weight forms, and τ is used for half-integer weight forms. Recall that we write $q=e^{2\pi iz}$ and $q=e^{2\pi i\tau}$. Let L be the lattice defined in Section 1.2.1, $\Delta\in\mathbb{Z}$ be a fundamental discriminant, and $r\in\mathbb{Z}$ such that $\Delta\equiv r^2 \pmod{4N}$. By ρ we denote the Weil representation associated to the lattice L.

5.1. Orthogonality to cusp forms

In this section we show that the Kudla-Millson and the Bruinier-Funke lift are orthogonal to cusp forms with respect to the regularized Petersson inner product. Recall that for $g\in S_{3/2+k,\widetilde{\rho}}$ we have

$$\left(\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F),g(\tau)\right)^{\mathrm{reg}}_{3/2+k,\widetilde{\rho}}=\lim_{t\to\infty}\int_{\mathcal{F}_t}\left\langle\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F),g(\tau)\right\rangle v^{3/2+k}d\mu(\tau),$$

where $\mathcal{F}_t$ denotes the truncated fundamental domain $\mathcal{F}_t=\{\tau\in\mathbb{H}\,:\,\Im(\tau)\le t\}$.

Theorem 5.1.1. *For $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)\in M^!_{3/2+k,\widetilde{\rho}}$, where $k\geq 0$, and $g\in S_{3/2+k,\widetilde{\rho}}$ we have*

$$\left(\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F),g(\tau)\right)^{reg}_{3/2+k,\widetilde{\rho}}=0.$$

The same holds for $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)\in M^!_{3/2+k,\widetilde{\rho}}$, where $k\geq 0$, and $g\in S_{3/2+k,\widetilde{\rho}}$, i.e.

$$\left(\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F),g(\tau)\right)^{reg}_{3/2+k,\widetilde{\rho}}=0.$$

Proof of Theorem 5.1.1. We only prove the statement for the Kudla-Millson theta lift since the arguments carry over directly for the Bruinier-Funke lift. To simplify notation we prove the theorem in the untwisted case. Since the twisted lift is essentially a linear combination of untwisted ones the arguments carry over directly (see the proof of Theorem 3.3.1).

Using the dominated convergence theorem it is tedious but straightforward to show that interchanging the integration with respect to z and τ is allowed. That is

$$\begin{aligned}
&\lim_{t\to\infty}\int_{\mathcal{F}_t}\left\langle\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F),g(\tau)\right\rangle v^{3/2+k}d\mu(\tau)\\
&=\lim_{t\to\infty}\int_{\mathcal{F}_t}\left\langle R^{k/2}_{3/2,\tau}\int_M(R^k_{-2k,z}F)(z)\Theta(\tau,z,\varphi_{\mathrm{KM}}),g(\tau)\right\rangle v^{3/2+k}d\mu(\tau)\\
&=\int_M(R^k_{-2k,z}F)(z)\lim_{t\to\infty}\int_{\mathcal{F}_t}\left\langle R^{k/2}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}}),g(\tau)\right\rangle v^{3/2+k}d\mu(\tau).
\end{aligned}$$

We consider the cases $k=0$ and $k>0$ separately.

Proof for $k>0$: We first show that for $k>0$

$$\lim_{t\to\infty}\int_{\mathcal{F}_t}\left\langle R^{k/2}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}}),g(\tau)\right\rangle v^{3/2+k}d\mu(\tau)=0.$$

Following the proof of Theorem 4.1 in [BOR08] we let

$$H:=v^{k-1/2}\overline{R^{k/2-1}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}})}$$

and

$$h:=R^{k/2}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}})=v^{-k-3/2}\overline{L_{1/2-k,\tau}H}.$$

Note that $R^{k/2-1}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}})$ is only defined for $k>0$. We obtain

$$\int_{\mathcal{F}_t}\left\langle R^{k/2}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}}),g(\tau)\right\rangle v^{3/2+k}d\mu(\tau)=\int_{\mathcal{F}_t}\left\langle v^{-k-3/2}\overline{L_{1/2-k,\tau}H},g(\tau)\right\rangle v^{3/2+k}d\mu(\tau).$$

We have that (see also Lemma 2.3.11)

$$\begin{aligned}\left\langle v^{-k-3/2}\overline{L_{1/2-k,\tau}H}, g(\tau)\right\rangle v^{3/2+k}d\mu(\tau) &= \left\langle \overline{2iv^2\frac{\partial}{\partial\overline{\tau}}H}, g(\tau)\right\rangle \frac{dudv}{v^2}\\ &= -\left\langle \overline{\frac{\partial}{\partial\overline{\tau}}H(\tau)}, g(\tau)\right\rangle d\tau d\overline{\tau}. \qquad (5.1.1)\end{aligned}$$

By the holomorphicity of g we obtain that (5.1.1) equals

$$-\left\langle \overline{\frac{\partial}{\partial\overline{\tau}}H(\tau)}, g(\tau)\right\rangle d\tau d\overline{\tau} = -\partial\left(\left\langle \overline{H(\tau)}, g(\tau)\right\rangle d\overline{\tau}\right) = -d\left(\left\langle \overline{H(\tau)}, g(\tau)\right\rangle d\overline{\tau}\right).$$

We now apply Stoke's Theorem. Since the integrand is $\mathrm{SL}_2(\mathbb{Z})$-invariant the equivalent pieces of the boundary of the fundamental domain cancel and we obtain

$$\begin{aligned}&\int_{\mathcal{F}_t}\left\langle R^{k/2}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}}), g(\tau)\right\rangle v^{3/2+k}d\mu(\tau)\\ &= -\int_{\partial\mathcal{F}_t}\left\langle v^{k-1/2}R^{k/2-1}_{3/2,\tau}\Theta(\tau,z,\varphi_{\mathrm{KM}}), g(\tau)\right\rangle d\tau\\ &= \sum_{h\in L'/L}\int_{-1/2}^{1/2} t^{k-1/2}R^{k/2-1}_{3/2,\tau}\theta_h(u+it,z,\varphi_{\mathrm{KM}})\overline{g_h(u+it)}du. \qquad (5.1.2)\end{aligned}$$

Plugging in the Fourier expansions of the two series and carrying out the integration over u we see that (5.1.2) equals

$$\sum_{h\in L'/L} t^{k-1/2}\sum_{n=1}^{\infty} b(n,h)a(n,h)e^{-4\pi nt},$$

where $b(n,h)$ and $a(n,h)$ denote the Fourier coefficients of g_h and $R^{k/2-1}_{3/2,\tau}\theta_h$ respectively. By classical results these coefficients grow very moderately and thus, the main contribution comes from the exponential terms, implying that the limit tends to 0 as $t\to\infty$.

Proof for $k=0$: For $k=0$ we use an argument for harmonic forms on Riemann surfaces to show that

$$\lim_{t\to\infty}\int_{\mathcal{F}_t}\theta_h(\tau,z,\varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau) = 0,$$

where we consider the components of the Petersson inner product separately now.

We first show that $\Delta_{0,z}$ annihilates this expression. Since the partial derivatives $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ of $\int_{\mathcal{F}_t}\theta_h(\tau,z,\varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau)$ converge locally uniformly in z as $t\to\infty$, we can interchange differentiation and the limit.

Recall that we have $\Delta_{3/2,\tau}\Theta(\tau, z, \varphi_{\mathrm{KM}}) = \frac{1}{4}\Delta_{0,z}\Theta(\tau, z, \varphi_{\mathrm{KM}})$ by (2.6.10), which implies

$$\int_{\mathcal{F}_t} \Delta_{0,z}\theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau)$$
$$= 4\int_{\mathcal{F}_t} \Delta_{3/2,\tau}\theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau).$$

By Lemma 4.3 of [Bru02] we find

$$\int_{\mathcal{F}_t} \Delta_{3/2,\tau}\theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau)$$
$$= \int_{\mathcal{F}_t} \theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{\Delta_{3/2,\tau}g_h(\tau)}v^{3/2}d\mu(\tau) \tag{5.1.3}$$
$$+ \int_{-1/2}^{1/2} \left[\theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{L_{3/2,\tau}g_h(\tau)}v^{3/2}\right]_{v=t} du \tag{5.1.4}$$
$$- \int_{-1/2}^{1/2} \left[L_{3/2,\tau}\theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}\right]_{v=t} du. \tag{5.1.5}$$

The holomorphicity of g implies that the integrals in (5.1.3) and (5.1.4) vanish. When plugging in the Fourier expansions of $g_h(u+it)$ and $L_{3/2,\tau}\theta_h(u+it, z, \varphi_{\mathrm{KM}})$ and integrating over u we see that the resulting expression is exponentially decaying as $t \to \infty$, which then implies

$$\Delta_{0,z} \lim_{t\to\infty} \int_{\mathcal{F}_t} \theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau) = 0.$$

Writing $\lim_{t\to\infty} \int_{\mathcal{F}_t} \theta_h(\tau, z, \varphi_{\mathrm{KM}})\overline{g_h(\tau)}v^{3/2}d\mu(\tau) = h(z)dz \wedge d\bar{z}$ for a smooth function on M, we have $\Delta_{0,z}h(z) = 0$. By the square-exponential decay of the Kudla-Millson theta function (see Proposition 2.6.9) $\Delta_{0,z}h(z) = 0$ implies that $h(z)$ is constant [Bru02, Corollary 4.22]. So it remains to show that this constant is zero. We do this by showing that for $z \in \mathbb{H}$ and σ_ℓ as in Section 1.2.3 we have

$$\lim_{y\to i\infty} h(\sigma_\ell z) = 0. \tag{5.1.6}$$

For simplicity, we only consider the cusp $\ell = \infty$. A careful analysis yields that we can interchange the limit processes with respect to t and y. The square exponential decay of $\theta_h(\tau, z, \varphi_{\mathrm{KM}})$ implies that $\lim_{y\to i\infty} \theta_h(\tau, z, \varphi_{\mathrm{KM}}) = 0$. Therefore, the limit $\lim_{y\to i\infty} h(y)$ vanishes. $\square$

5.2. Duality and Hecke action

The orthogonality of the two lifts together with Proposition 2.3.20 and Remark 2.3.16 directly implies the following duality results.

Corollary 5.2.1. *Let $\kappa = 3/2 + k$ if k is odd and $\kappa = 1/2 - k$ if k is even. We let f be a harmonic weak Maass form of weight κ transforming with representation $\overline{\widetilde{\rho}}$ and denote the (m,h)-th Fourier coefficient of the holomorphic part by $c_f^+(m,h)$. Moreover, let $F \in M^!_{-2k}(N)$, such that $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ is weakly holomorphic of weight $2-\kappa$ and transforms with representation $\widetilde{\rho}$. We denote the Fourier coefficients of $\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ by $a^+_{\mathcal{I}^{\mathrm{KM}}}(m,h)$. Then we have*

$$\sum_{h\in L'/L} \sum_{\substack{m\geq 0\\ m\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} c_f^+(-m,h)a^+_{\mathcal{I}^{\mathrm{KM}}}(m,h) = -\sum_{h\in L'/L} \sum_{\substack{m\geq 0\\ -N|\Delta|m^2\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} c_f^+(N\,|\Delta|\,m^2,h)a^+_{\mathcal{I}^{\mathrm{KM}}}(-N\,|\Delta|\,m^2,h).$$

The same result holds for the Bruinier-Funke lift.

Corollary 5.2.2. *Let $\kappa = 3/2 + k$ if k is even and $\kappa = 1/2 - k$ if k is odd. We let f be a harmonic weak Maass form of weight κ transforming with representation $\overline{\widetilde{\rho}}$ and denote the (m,h)-th Fourier coefficient of the holomorphic part by $c_f^+(m,h)$. Moreover, let $F \in M^!_{-2k}(N)$, such that $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)$ is weakly holomorphic of weight $2-\kappa$ and transforms with representation $\widetilde{\rho}$. We denote the Fourier coefficients of the lift $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)$ by $a^+_{\mathcal{I}^{\mathrm{BF}}}(m,h)$. Then we have*

$$\sum_{h\in L'/L} \sum_{\substack{m\geq 0\\ m\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} c_f^+(-m,h)a^+_{\mathcal{I}^{\mathrm{BF}}}(m,h) = -\sum_{h\in L'/L} \sum_{\substack{m\geq 0\\ -N|\Delta|m^2\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} c_f^+(N\,|\Delta|\,m^2,h)a^+_{\mathcal{I}^{\mathrm{BF}}}(-N\,|\Delta|\,m^2,h).$$

Remark 5.2.3. These results can be rephrased in terms of the two lifts, i.e. choosing $f = \mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,F)$ in Corollary 5.2.1, respectively $f = \mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F)$ in Corollary 5.2.2, we obtain a duality result for the two lifts. With these formulas we can recover duality results between Poincaré series as in [Zag02, BO07] (also see the introduction).

Using the action of the Hecke algebra we also obtain formulas of such type for a wider class of coefficients.

Proposition 5.2.4. *Let the hypotheses be as in Corollary 5.2.1. Then we have*

$$\sum_{h\in L'/L} \sum_{\substack{m\geq 0\\ m\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} c_f^{+,*}(-m,h)\mathbf{t}_{\Delta,r}(F;m,h) = -\sum_{h\in L'/L} \sum_{\substack{m\geq 0\\ -N|\Delta|m^2\equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})}} c_f^{+,*}(N\,|\Delta|\,m^2,h)a^+_{\mathcal{I}^{\mathrm{KM}}}(-N\,|\Delta|\,m^2,h),$$

where $c_f^{+,*}(n,h)$ *is as in* (2.3.10).

Remark 5.2.5. An analogous result holds for the Bruinier-Funke lift.

Proof. We can assume that $g \in S^{\mathrm{new}}_{2-\kappa,\widetilde{\rho}}$ such that $\xi_\kappa(f) = g/||g||^2$. Then we have

$$\begin{aligned}\left\{\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F), f|_\kappa T(\ell)\right\} &= \left(\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F), \xi_\kappa\left[(f|_\kappa T(\ell))(\tau)\right]\right)^{\mathrm{reg}}_{2-\kappa,\widetilde{\rho}} \\ &= \left(\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F), \ell^{2\kappa-2}\left(\xi_\kappa(f)|_{2-\kappa}T(\ell)\right)(\tau)\right)^{\mathrm{reg}}_{2-\kappa,\widetilde{\rho}}\end{aligned}$$

by equation (7.2) in [BO10]. This equals

$$\left(\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F), \ell^{2\kappa-2}\frac{1}{||g||^2}g|_\kappa T(\ell)\right)^{\mathrm{reg}}_{2-\kappa,\widetilde{\rho}}.$$

Since g is an eigenform we obtain

$$\lambda_\ell \ell^{2\kappa-2}\left(\mathcal{I}^{\mathrm{KM}}_{\Delta,r}(\tau,F), \frac{g}{||g||^2}\right)^{\mathrm{reg}}_{2-\kappa,\widetilde{\rho}},$$

where λ_ℓ is the eigenvalue of g under $T(\ell)$. But this vanishes since the lift is orthogonal to cusp forms. $\square$

6. Elliptic curves and harmonic Maass forms

In this chapter we study the relation between the central value and central derivative of the Hasse-Weil zeta function of twists of an elliptic curve E over $\mathbb{Q}$ and the Fourier coefficients of a harmonic Maass form of weight $1/2$ associated to E.

The starting point is a theorem by Bruinier and Ono [BO10, Theorem 7.8]. They consider weight $1/2$ harmonic Maass forms f whose image under $\xi_{1/2}$ is equal to a real multiple of a weight $3/2$ newform g that maps to G under the Shimura correspondence. That is, we have the following picture

$$\begin{tikzcd} & G \in S_2(N) \\ f \in H^+_{1/2,\widetilde{\rho}} \arrow[r, "\xi_{1/2}"] & g \in S_{3/2,\bar{\widetilde{\rho}}}. \arrow[u, "\text{Shimura}"'] \end{tikzcd} \tag{6.0.1}$$

Employing deep work of Shimura and Waldspurger they proved that the Fourier coefficients of the non-holomorphic part of f as above give exact formulas for $L(G, D, 1)$, where D is an appropriate discriminant. Using the theory of Borcherds products and the Gross–Zagier Theorem they show that at the same time the coefficients of the holomorphic part of f encode the vanishing of the central derivatives $L'(G, D', 1)$ (where D' is again an appropriate discriminant). However, the harmonic Maass form f does not directly arise from the cusp form G.

In this chapter we show that we can complete the diagram (6.0.1) by constructing a canonical preimage $\mathcal{W}_E$ of a newform $G_E \in S_2^{\text{new}}(N_E)$ associated to an elliptic curve E over $\mathbb{Q}$ and employing the Bruinier-Funke lift. We obtain the following commutative diagram

$$\begin{tikzcd} \mathcal{W}_E \in H_0(N_E) \arrow[r, "\xi_0"] \arrow[d, "\mathcal{I}^{\text{BF}}_{\Delta,r}"] & G_E \in S_2^{\text{new}}(N_E) \arrow[d, "\text{Shintani}"] \\ \mathcal{I}^{\text{BF}}_{\Delta,r}(\tau, \mathcal{W}_E) \in H^+_{1/2,\widetilde{\rho}} \arrow[r, "\xi_{1/2}"] & g \in S_{3/2,\bar{\widetilde{\rho}}}. \end{tikzcd} \tag{6.0.2}$$

Recall that for an elliptic curve $E:\ y^2 = x^3 + ax + b$ over $\mathbb{Q}$ and a fundamental discriminant d, the d-quadratic twist of E is defined by $E_d:\ dy^2 = x^3 + ax + b$.

The above diagram leads to a connection between the vanishing of $L(E_d, 1)$ and the vanishing of the d-th coefficient of the non-holomorphic part of $\mathcal{I}^{\text{BF}}_{\Delta,r}(\tau, \mathcal{W}_E)$. Here, $d \neq 1$ is a fundamental discriminant that we choose depending on the discriminant Δ which is used to twist the Bruinier-Funke lift.

Moreover, we prove that the algebraicity of the d-th Fourier coefficient of the holomorphic part of the Bruinier-Funke lift of $\mathcal{W}_E$ as above encodes the vanishing of $L'(E_d, 1)$ (again d depends on Δ). Note that our proof is independent of the results in [BO10]. Moreover, it also applies to the coefficients of the Kudla-Millson lift of $\mathcal{W}_E$ (or more generally a harmonic Maass form of weight 0 that maps to a newform of weight 2 under ξ_0).

In the first part of the chapter we briefly introduce the theory of elliptic curves and explain the connection between L-functions of weight 2 cusp forms and of elliptic curves. Moreover, we discuss the notion of differentials of the first, second and third kind on a complex Riemann surface. We then turn to the construction of the harmonic Maass form $\mathcal{W}_E$ associated to an elliptic curve E. The last part of the chapter is devoted to the proof of the relation between the Fourier coefficients of $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \mathcal{W}_E)$ and the central values and derivatives of the Hasse-Weil zeta function of E. Moreover, we present some implications of this result for periods of differentials of the first and second kind associated to $\mathcal{W}_E$.

6.1. Elliptic curves and modular forms

In this section we summarize some results on elliptic curves and their relations to modular forms. Most of the material is taken from [DS05, Sil09].

6.1.1. Elliptic curves and Weierstrass functions

Let E be an elliptic curve over $\mathbb{Q}$, i.e. a non-singular curve defined by an equation of the form

$$y^2 = x^3 + ax + b$$

with a, $b \in \mathbb{Q}$. The condition that E is non-singular is equivalent to the non-vanishing of the discriminant $\Delta = -16(4a^3 + 27b^2)$.

Over $\mathbb{C}$ every elliptic curve is isomorphic to $\mathbb{C}/\Lambda_E$, where Λ_E is a certain lattice in $\mathbb{C}$. To define the corresponding isomorphism we introduce the Weierstrass $\wp$-function. For a lattice Λ in $\mathbb{C}$ and $z \in \mathbb{C} \setminus \Lambda$ we let

$$\wp(\Lambda; z) := \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

The Weierstrass $\wp$-function converges absolutely and uniformly on compact subsets of $\mathbb{C}$ away from Λ. It is an even Λ-invariant function and is holomorphic on $\mathbb{C} \setminus \Lambda$. In points of Λ it has poles of second order with residue 0.

For an integer $k > 0$ we let $G_{2k}(\Lambda)$ be the standard weight $2k$ Eisenstein series defined as

$$G_{2k}(\Lambda) := \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-2k}.$$

This series converges for $k > 1$. For $k = 1$ we will consider its analytic continuation (see (6.2.3)).

We obtain the following isomorphism between E and $\mathbb{C}/\Lambda_E$.

Proposition 6.1.1. *Let E be an elliptic curve given by the equation*

$$E(\Lambda_E) : y^2 = 4x^3 - 60G_4(\Lambda_E) - 140G_6(\Lambda_E)$$

with Λ_E a lattice in $\mathbb{C}$. The map

$$\begin{aligned} \Psi : (\mathbb{C}/\Lambda_E) \setminus \{\Lambda_E\} &\to E(\Lambda_E) \\ z &\mapsto (\wp(\Lambda_E; z), \wp'(\Lambda_E; z)) \end{aligned}$$

is an isomorphism.

The Weierstrass $\wp$-function has a Laurent expansion of the form

$$\wp(\Lambda; z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}(\Lambda)z^{2n-2}$$

for all z satisfying $0 < |z| < \inf\{|w| : w \in \Lambda \setminus \{0\}\}$.

We are also interested in the Weierstrass ζ-function

$$\zeta(\Lambda; z) := \frac{1}{z} + \sum_{w \in \Lambda\setminus\{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right) = \frac{1}{z} - \sum_{k=1}^{\infty} G_{2k+2}(\Lambda)z^{2k+1}. \tag{6.1.1}$$

It satisfies

$$\wp(\Lambda; z) = -\zeta'(\Lambda; z).$$

6.1.2. Modularity of elliptic curves

Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$ with conductor N_E (an integer divisible by the same prime numbers as the discriminant of E). Let $s \in \mathbb{C}$ with $\Re(s) > 1$ and define the L-function (or Hasse-Weil zeta function) of E by

$$L(E, s) = \prod_{p \text{ prime}} L_p(E, s),$$

where

$$L_p(E, s) = \begin{cases} (1 - a(p)p^{-s} + p^{1-2s})^{-1} & p \nmid N_E, \\ (1 - a(p)p^{-s})^{-1} & p|N_E,\ p^2 \nmid N_E, \\ 1 & p^2|N_E. \end{cases}$$

Here $a(p) = p + 1 - (\#\text{of points of } E \text{ modulo } p)$.

In the mid-1950's Taniyama conjectured that the L-function of an elliptic curve agrees with the L-function of a weight 2 newform that has eigenvalues $a(p)$ under the action of the Hecke operators. One direction of this conjecture was proved in the 1960's by Eichler and Shimura, who showed that given a Hecke eigenform G in $S_2(N)$ with integral Fourier coefficients, there is an elliptic curve $E/\mathbb{Q}$ such that $L(E,s) = L(G,s)$.

The other direction was finally proved in the 1990's by Wiles, Breuil, Conrad, Diamond, and Taylor [Wil95, BCDT01] (see [DS05] for an overview of the different versions of the modularity theorem and [CSS97] for an overview of the proof of Fermat's last theorem involving the proof of the modularity of elliptic curves).

Theorem 6.1.2 (Modularity Theorem). *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E. There is a weight 2 newform $G_E(\tau) = \sum_{n=1}^{\infty} a_E(n)q^n \in S_2(N_E)$ that satisfies*

$$L(G_E, s) = L(E, s),\ s \in \mathbb{C}.$$

Remark 6.1.3. Let the notation be as above. The theorem implies that all the properties of $L(G_E, s)$ also apply to $L(E,s)$, i.e. it can be holomorphically continued to $\mathbb{C}$, it can be written as

$$L(E,s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}.$$

and it satisfies a functional equation in s as in Proposition 2.1.7.

A different version of the modularity theorem above is the following theorem (stated as Theorem 2.5.1 in [DS05]).

Theorem 6.1.4. *Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E. There is a surjective holomorphic function of complex Riemann surfaces from the modular curve $X_0(N_E)$ to the elliptic curve E of conductor N_E,*

$$\phi_E : X_0(N_E) \to E.$$

This function is called the modular parametrization of E.

6.1.3. The Birch and Swinnerton-Dyer Conjecture

Throughout let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E over $\mathbb{Q}$. We consider the group $E(\mathbb{Q})$ of rational points on E. Mordell proved the following famous theorem.

Theorem 6.1.5 (Mordell-Weil). *We have*

$$E(\mathbb{Q}) = E(\mathbb{Q})^{tors} \oplus \mathbb{Z}^r$$

for some integer $r \geq 0$ called the rank of E. Here, $E(\mathbb{Q})^{tors}$ is a finite abelian group.

The Birch and Swinnerton-Dyer Conjecture relates the rank r to the analytic properties of the L-function of the elliptic curve. It is one of the most famous open problems in number theory. It was named after Bryan Birch and Peter Swinnerton-Dyer who came up with the conjecture in the 1960's with the help of extensive computer-based computations. We state the original version of the conjecture.

Conjecture 6.1.6. The Taylor expansion of $L(E,s)$ at $s=1$ is of the form

$$L(E,s) = c\cdot(s-1)^r + \textit{higher order terms}$$

with $c \neq 0$ and $r = \operatorname{rank}(E)$.

There is a more detailed version of this conjecture predicting the value of $L^{(r)}(E,1)$, namely

$$L^{(r)}(E,1) = c\cdot\Omega\cdot R.$$

Here, $\Omega = \int_{E(\mathbb{R})} dx/y$ is the real period, and R is the regulator, the determinant of a $r\times r$-matrix whose entries are given by a height pairing applied to a system of generators of $E(\mathbb{Q})/E(\mathbb{Q})^{\mathrm{tors}}$. The number c is a non-zero rational number involving the order of the Tate-Shafarevich group and several other expressions (see for example [KZ01, Wil06]).

The conjecture has only been proved in the case that the analytic rank is equal to 0 or 1 in [GZ86] and [Kol88].

Theorem 6.1.7 (Gross-Zagier, Kolyvagin)**.** *If $L(E,s) = c(s-1)^r +$ higher order terms and $r = 0$ or 1, then the Birch and Swinnerton-Dyer conjecture is true.*

The remaining cases are still unknown, however, by recent results of Bhargava and Shankar combined with results of Nekovà, the Dokchitser brothers and Skinner and Urban it is known that a positive proportion of elliptic curves over $\mathbb{Q}$ has analytic rank zero and thus satisfies the Birch and Swinnerton-Dyer conjecture [BS, Nek01, DD10, SU14].

An interesting question is if one can compute $L(E,s)$ and $L'(E,s)$ at $s=1$. In fact, this was done by Gross and Zagier in their part of the proof of Theorem 6.1.7. Roughly, they proved (if $L(E,1)=0$) that $L'(E,1)$ is given by a multiple of the height of a Heegner point (see [GZ86] and (6.3.7)). Here, we recall results of Waldspurger as well as Kohnen and Zagier saying that the coefficients of half-integral weight cusp forms are essentially the square roots of central values of quadratic twists of modular L-functions.

In order to state these results, we first define quadratic twists of elliptic curves. For an elliptic curve $E:\, y^2 = x^3+ax+b$ over $\mathbb{Q}$ and a fundamental discriminant Δ, we consider the Δ-quadratic twist of E

$$E_\Delta :\, \Delta y^2 = x^3+ax+b. \tag{6.1.2}$$

The corresponding twisted L-function is given by

$$L(E_\Delta, s) = \sum_{n=1}^{\infty} \left(\frac{\Delta}{n}\right) a(n)n^{-s},$$

and this corresponds to the twisted (modular) L-function $L(G_E,\Delta,s)$ with G_E as in Theorem 6.1.2.

Building upon results of Waldspurger [Wal81] Kohnen and Zagier proved the following theorem in [KZ81].

Theorem 6.1.8 (Waldspurger, Kohnen–Zagier). *Let $G \in S_{2k}(1)$ be a normalized Hecke eigenform and let $g \in S^+_{k+1/2}(4)$ (the Kohnen plus-space) be the form that corresponds to G under the Shimura correspondence. We denote the n-th coefficient of g by $c(n)$. Let D be a fundamental discriminant with $(-1)^k D > 0$. Then*

$$\frac{|c(|D|)|^2}{||g||^2} = \frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(G,D,k)}{||G||^2}\,.$$

6.1.4. Differentials on Riemann surfaces, modular and elliptic curves

In this section we introduce the identification between modular forms and differential forms on $X_0(N)$ following the exposition in [Sil94, DS05]. Moreover, we define the notion of differentials of the first, second and third kind on compact Riemann surfaces and present a theorem by Scholl on the transcendence of differentials of the third kind.

By $\mathcal{M}_k(N)$ we denote the space of meromorphic modular forms $f : \mathbb{H} \to \mathbb{C}$. These forms satisfy the following conditions:

(i) $(f|_k\gamma)(z) = f(z)$ for all $\gamma \in \Gamma_0(N)$.

(ii) f is meromorphic on $\mathbb{H}$.

(iii) f is meromorphic at the cusps of $\Gamma_0(N)$.

Let X be a compact Riemann surface. By Ω_X we denote the $\mathbb{C}(X)$-vector space of meromorphic differential 1-forms on X. The space of meromorphic differentials of degree k is the k-fold tensor product

$$\Omega_X^k = \Omega_X^{\otimes k}.$$

We have the following identification between meromorphic differentials and meromorphic modular forms.

Proposition 6.1.9 (Proposition 3.7 in Chapter 1 of [Sil94] and Theorem 3.3.1 in Chapter 3 of [DS05]). *Let $f \in \mathcal{M}_{2k}(N)$. The k-form $f(\tau)(d\tau)^k$ on $\mathbb{H}$ descends to give a meromorphic k-form w_f on $X_0(N)$. That is, there is a k-form $w_f \in \Omega^k_{X_0(N)}$ such that*

$$\phi^*(w_f) = f(\tau)(d\tau)^k,$$

where $\phi : \mathbb{H} \to X_0(N)$ is the usual projection.

Next we introduce the notion of differentials of the first, second, and third kind. A differential of the first kind on X is a holomorphic 1-form. A differential of the second kind is a meromorphic 1-form on X whose residues all vanish. A differential of the third kind on X is a meromorphic 1-form on X whose poles are all of first order with residues in $\mathbb{Z}$. Later we will relax the condition on the integrality of the residues.

Let ψ be a differential of the third kind on X that has poles at the points P_j with residues a_j, and is holomorphic elsewhere. The residue divisor of ψ is defined by

$$\mathrm{res}(\psi) := \sum_j a_j P_j.$$

The restriction of $\mathrm{res}(\psi)$ to any component of X has degree 0. Conversely, if $D = \sum c_j P_j$ is a divisor on X whose restriction to any component has degree 0, then the Riemann-Roch theorem and Serre duality imply that there is a differential of the third kind ψ_D with $\mathrm{res}(\psi_D) = D$ (see for example [GH94, p. 233]). The differential ψ_D is determined by this condition up to addition of a differential of the first kind.

Using the Riemann period relations, one can show that there is a unique differential of the third kind η_D on X with residue divisor D such that

$$\Re\left(\int_\gamma \eta_D\right) = 0$$

for all $\gamma \in H_1(X\setminus\{P_j\}, \mathbb{Z})$. It is called the *canonical differential of the third kind* associated with D.

A different characterization of η_D is given by Scholl.

Proposition 6.1.10 (Proposition 1 in [Sch86])**.** *The differential η_D is the unique differential of the third kind with residue divisor D which can be written as $\eta_D = \partial h_D$, where h_D is a harmonic function on $X \setminus \{P_j\}$.*

Remark 6.1.11. By Corollary 8.2 (ii) of [Spr57] we have (in the setting of Proposition 6.1.10)

$$h_D(z) = c_j \log|z - P_j| + H(z)$$

for a local variable z near P_j and a smooth function $H(z)$.

Example 6.1.12. (i) Let $G \in S_2(N)$ be a cusp form of weight 2 for $\Gamma_0(N)$. Then

$$\omega_G = 2\pi i\, G(z)dz \tag{6.1.3}$$

is a differential 1-form of the first kind on $X_0(N)$.

(ii) Let $F \in M_2^!(N)$ with vanishing constant coefficient. Then

$$2\pi i F(z)dz \tag{6.1.4}$$

is a differential 1-form of the second kind on $X_0(N)$.

(iii) Let D be a degree 0 divisor on $X_0(N)$ that is coprime to the cusps. Let $\mathcal{D}(z)$ be a meromorphic modular form of weight 2 for $\Gamma_0(N)$ whose poles lie on $D \subset Y_0(N)$ and are of first order with residues in $\mathbb{Z}$. Then

$$2\pi i\mathcal{D}(z)dz \tag{6.1.5}$$

is a differential of the third kind with residue divisor D on $X_0(N)$. The constant coefficient of the Fourier expansion of $\mathcal{D}(z) = \sum_{n=0}^{\infty} d(n)q^n$ at ∞ is the residue of η_D at ∞. We have analogous expansions at the other cusps of $\Gamma_0(N)$. We refer to $d(n)$ as the n-th coefficient of η_D at the cusp ∞.

Remark 6.1.13. For a geometric approach to harmonic Maass forms see [BF04, Proof of Theorem 3.7] and the work of Candelori [Can14]. Using different descriptions of the de Rham cohomology attached to modular forms one can show that there are differentials of the first and second kind, denoted by ω and ϕ, such that $\phi - \overline{\omega}$ is exact. For harmonic Maass forms of weight 0 this is reflected in Lemma 2.3.14.

Scholl proved an interesting criterion on the transcendence of differentials of the third kind. From now on we assume that X is defined over $\mathbb{Q}$. By $\bar{\mathbb{Q}}$ we denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We assume that D is a degree 0 divisor on X which is defined over a number field $F \subset \bar{\mathbb{Q}}$. Using results of Waldschmidt [Wal87] Scholl proved the following on the transcendence of a canonical differential of the third kind η_D with residue divisor D [Sch86].

Theorem 6.1.14 (Scholl)**.** *If some non-zero multiple of D is a principal divisor, then η_D is defined over F. Otherwise, η_D is not defined over $\bar{\mathbb{Q}}$.*

The q-expansion principle directly implies the following corollary if we take $X = X_0(N)$ (see also [BO10, Theorem 3.3]).

Corollary 6.1.15. *If some non-zero multiple of D is a principal divisor, then all the coefficients $d(n)$ of η_D at the cusp ∞ are contained in F. Otherwise, there is an n such that $d(n)$ is transcendental.*

We now describe differentials of the first and second kind on an elliptic curve. Let E be an elliptic curve of conductor N_E over $\mathbb{Q}$ and let $\omega_E = \frac{dx}{y}$ be the Néron differential of E. By $G_E \in S_2(N_E)$ we denote the cusp form corresponding to E under the modular parametrization. Multiplicity one implies that the pullback of ω_E under the modular parametrization is given by

$$\phi^*(\omega_E) = c_E \, \omega_{G_E},$$

where c_E denotes the Manin constant (see for example [Cre97]).

Example 6.1.16. (i) The Néron differential $\omega_E = \frac{dx}{y}$ is of the first kind. Its pullback under the complex uniformization $\mathbb{C}/\Lambda_E$ of E equals dz.

(ii) The differential

$$x \, \omega_E.$$

is of the second kind. The pullback of $x \, \omega_E$ under the complex uniformization $\mathbb{C}/\Lambda_E$ of E is equal to $\wp(z)dz$.

6.2. Canonical Weierstrass harmonic Maass forms

In this section we show that there are canonical preimages of weight 2 newforms G_E for the group $\Gamma_0(N_E)$ related to an elliptic curve E over $\mathbb{Q}$ of conductor N_E. This is joint work with Michael Griffin, Ken Ono, and Larry Rolen [AGOR] and is based on an idea of Pavel Guerzhoy [Gue13, Gue]. Note that the results in this section were worked out by Michael Griffin, Ken Ono, and Larry Rolen.

We write $G_E = \sum_{n=1}^{\infty} a_E(n)q^n$ for the Fourier expansion of G_E and define the Eichler integral $\mathcal{E}_E(z)$ of G_E by

$$\mathcal{E}_E(z) := -2\pi i \int_z^{i\infty} G_E(\tau)d\tau = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} \cdot q^n. \tag{6.2.1}$$

Recall the Weierstrass ζ-function defined in (6.1.1)

$$\zeta(\Lambda_E; z) := \frac{1}{z} + \sum_{w\in\Lambda_E\setminus\{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right) = \frac{1}{z} - \sum_{k=1}^{\infty} G_{2k+2}(\Lambda_E)z^{2k+1}.$$

It is not lattice invariant, but Eisenstein [Wei76] observed that a suitable modification is. Namely, he considered

$$\zeta^*(\Lambda_E; z) = \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\pi}{a(\Lambda_E)}\overline{z}, \tag{6.2.2}$$

where

$$S(\Lambda_E) := \lim_{s\to 0^+} \sum_{w\in\Lambda_E\setminus\{0\}} \frac{1}{w^2|w|^{2s}} \tag{6.2.3}$$

and $a(\Lambda_E)$ is the area of the fundamental parallelogram for Λ_E. For a new and short proof of the lattice invariance of ζ^* by Zagier see the first chapter of the (not yet finished) book by Bringmann, Folsom and Ono [BFO]. Zagier [Zag85] and Cremona [Cre94] showed that we can express $a(\Lambda_E)$ in terms of the degree of the modular parametrization as follows

$$a(\Lambda_E) = \frac{4\pi^2||G_E||^2}{\deg(\phi_E)}.$$

Evaluating $\zeta^*(\Lambda_E, z)$ at the Eichler integral $\mathcal{E}_E(z)$ we obtain a modular object of weight 0. The Eichler integral is not modular, however its obstruction to modularity is easily characterized. The map $\Psi_E\colon \Gamma_0(N) \to \mathbb{C}$ given by

$$\Psi_E(\gamma) := \mathcal{E}_E(z) - \mathcal{E}_E(\gamma z)$$

is a homomorphism of groups. Its image in $\mathbb{C}$ turns out to be the lattice Λ_E as in Proposition 6.1.1. Hence, since $\zeta^*(\Lambda_E; z)$ is invariant on the lattice, the map $\zeta^*(\Lambda_E; \mathcal{E}_E(z))$ is modular of weight 0.

We define
$$\mathcal{W}_E^*(z) := \zeta(\Lambda_E; \mathcal{E}_E(z)) - S(\Lambda_E) - \frac{\deg(\phi_E)}{4\pi||G_E||^2}\overline{\mathcal{E}_E(z)}.$$
We write $\mathcal{W}_E^{*,+}(z)$ for the holomorphic part of $\mathcal{W}_E^*(z)$ (which is given by $\zeta^*(\Lambda_E, \mathcal{E}_E(z))$). This form satisfies the following properties.

Theorem 6.2.1. *Assume the notation and hypotheses above. The following are true:*

(1) The poles of the holomorphic part of $\mathcal{W}_E^{*,+}(z)$ are precisely those points z for which $\mathcal{E}_E(z) \in \Lambda_E$.

(2) If the holomorphic part of $\mathcal{W}_E^*(z)$ has poles in $\mathbb{H}$, then there is a canonical modular function $M_E(z)$ on $\Gamma_0(N_E)$ with algebraic coefficients for which $\mathcal{W}_E^{*,+}(z) - M_E(z)$ is holomorphic on $\mathbb{H}$.

(3) We have that $\mathcal{W}_E^*(z) - M_E(z)$ is a weight 0 harmonic Maass form on $\Gamma_0(N_E)$.

Definition 6.2.2. Assuming the notation above we define a *Weierstrass harmonic Maass form* for the cusp form G_E by
$$\mathcal{W}_E(z) := \mathcal{W}_E^*(z) - M_E(z) = \mathcal{W}_E^+(z) + \mathcal{W}_E^-(z), \tag{6.2.4}$$
where
$$\mathcal{W}_E^+(z) = \zeta(\Lambda_E; \mathcal{E}_E(z)) - S(\Lambda_E)\mathcal{E}_E(z) - M_E(z), \tag{6.2.5}$$
$$\mathcal{W}_E^-(z) = -\frac{\deg(\phi_E)}{4\pi||G_E||^2}\overline{\mathcal{E}_E(z)}. \tag{6.2.6}$$

Remark 6.2.3. Note that the choice we make for $\mathcal{W}_E(z)$ is not unique. Note also, that every such $\mathcal{W}_E(z)$ satisfies
$$\xi_0(\mathcal{W}_E(z)) = \frac{1}{\deg\phi_E}\frac{G_E}{||G_E||^2}.$$

Remark 6.2.4. In [AGOR] we referred to the holomorphic part of $\mathcal{W}_E^*(z)$ as the *Weierstrass mock modular form* for E. It is a simple task to compute this form. Using the two Eisenstein numbers $G_4(\Lambda_E)$ and $G_6(\Lambda_E)$, one then computes the remaining Eisenstein numbers using the recursion
$$G_{2n}(\Lambda_E) := \sum_{j=2}^{n-2} \frac{3(2j-1)(2n-2j-1)}{(2n+1)(2n-1)(n-3)} \cdot G_{2j}(\Lambda_E)G_{2n-2j}(\Lambda_E).$$
Armed with the Fourier expansion of $G_E(z)$ and $S(\Lambda_E)$, one then simply computes the functions in (6.2.1) and (6.2.5).

Proof. First note that we already proved the modularity (of weight 0) for $\mathcal{W}_E^*(z)$.

Part (1) of Theorem 6.2.1 follows by noting that $\zeta^*(\Lambda_E, z)$ diverges precisely for $z \in \Lambda_E$. This divergence must result from a pole in the holomorphic part, i.e. in $\zeta(\Lambda_E, z) - S(\Lambda_E)z$.

In order to establish part (2), we consider the modular function $\wp(\Lambda_E; \mathcal{E}_E(z))$. We observe that $\wp(\Lambda_E; \mathcal{E}_E(z))$ is meromorphic with poles precisely for those z such that $\mathcal{E}_E(z) \in \Lambda_E$. We claim that $\wp(\Lambda_E; \mathcal{E}_E(z))$ may be decomposed into modular functions with algebraic coefficients, each with only a simple pole at one such z and possibly at cusps. This follows from a careful inspection of the standard proof that $M_0^!(N) = \mathbb{C}\left(j(z), j(Nz)\right)$. For example, following the proof of Theorem 11.9 in [Cox11], one obtains an expression for the given modular function in terms of a function $G(z)$ and a modular function with rational coefficients. The function $G(z)$ clearly lies in $\overline{\mathbb{Q}}\left(j(z), j(Nz)\right)$ whenever we start with a modular function with algebraic coefficients at all cusps, from which the claim follows easily.

These simple modular functions may then be combined appropriately to construct the function $M_E(z)$ to cancel the poles of $\mathcal{W}_E^+(z)$, and the remainder of the proof of (3) then follows from straightforward calculations. $\square$

We also state the Fourier expansion of $\mathcal{W}_E^*(z)$ at cusps. Let N_E be square-free and recall that the Atkin-Lehner involutions act transitively on the cusps of $X_0(N_E)$. By Atkin-Lehner Theory, there is a $\lambda_Q \in \{\pm 1\}$ for which $G_E|_2 W_Q^{N_E} = \lambda_Q G_E$.

Theorem 6.2.5 (Theorem 1.2 in [AGOR])**.** *If Q is an exact divisor of N_E, then*

$$(\mathcal{W}_E^*|_0 W_Q^{N_E})(z) = \mathcal{W}_E^{*,+}(\lambda_Q(\mathcal{E}_E(z) - \Omega_Q(G_E))) - \frac{\deg(\phi_E)}{4\pi||G_E||^2} \cdot \overline{\lambda_Q(\mathcal{E}_E(z) - \Omega_Q(G_E))},$$

where we have

$$\Omega_Q(G_E) := -2\pi i \int_{(W_Q^{N_E})^{-1} i\infty}^{i\infty} G_E(z)dz.$$

Remark 6.2.6. In particular, we have $\Omega_Q(G_E) = L(G_E, 1)$. By the modular parametrization, we have that $\wp(\Lambda_E; \mathcal{E}_E(z))$ is a modular function on $\Gamma_0(N_E)$. We then have for each $Q|N_E$ that $\Omega_Q(G_E) \in r\Lambda_E$, where r is a rational number. This can be seen by considering the constant term of $\wp(\Lambda_E; \mathcal{E}_E(z))$ at cusps. The constant term of $\wp(\Lambda_E; \mathcal{E}_E(z))$ is $\wp(\Lambda_E; \Omega_Q(G_E))$. More generally, if N_E is square-free, then $\Omega_Q(G_E)$ maps to a rational torsion point of E.

Remark 6.2.7. The expansion of $\mathcal{W}_E^*(z)$ at cusps can be explicitly computed using the addition law for the Weierstrass ζ-function

$$\zeta(\Lambda_E; z_1 + z_2) = \zeta(\Lambda_E; z_1) + \zeta(\Lambda_E; z_2) + \frac{1}{2} \frac{\wp'(\Lambda_E; z_1) - \wp'(\Lambda_E; z_2)}{\wp(\Lambda_E; z_1) - \wp(\Lambda_E; z_2)}.$$

Remark 6.2.8. The function $\mathcal{W}_E^*(z)$ admits special p-adic properties under the action of the Hecke algebra that were also investigated in [AGOR, Theorem 1.3]. Namely, we have:

If $p \nmid N_E$ is ordinary, then there is a constant $\mathfrak{S}_E(p)$ for which

$$\lim_{n\to+\infty} \frac{\left[q\frac{d}{dq}\zeta(\Lambda_E;\mathcal{E}_E(z))\right]|T(p^n)}{a_E(p^n)} = \mathfrak{S}_E(p)G_E(z).$$

6.3. Lifts of Weierstrass harmonic Maass forms and Hasse-Weil zeta functions

In this section we study the connection between the vanishing of the central twisted L-values and L-derivatives of elliptic curves and the Fourier coefficients of the associated harmonic Maass form of weight 1/2.

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N_E and $G_E \in S_2^{\text{new}}(N_E)$ the newform corresponding to E as described in Theorem 6.1.2.

Recall the Weierstrass harmonic Maass form

$$\mathcal{W}_E(z) = \zeta(\Lambda_E;\mathcal{E}_E(z)) - S(\Lambda_E)\mathcal{E}_E(z) - M_E(z) - \frac{\deg(\phi_E)}{4\pi||G_E||^2}\overline{\mathcal{E}_E(z)}.$$

We assume that the principal parts of $\mathcal{W}_E$ at all cusps other than ∞ vanish. This can be obtained by choosing a suitable function $M_E(z)$ in Theorem 6.2.1. In particular, this choice of $\mathcal{W}_E$ implies that the constant coefficients of $\mathcal{W}_E$ vanish at all cusps of $\Gamma_0(N)$. Moreover, we normalize $\mathcal{W}_E$ such that it maps to $G_E/||G_E||^2$ under ξ_0. By slight abuse of notation we denote this form by $\mathcal{W}_E(z)$ again.

Let $\Delta \neq 1$ be a fundamental discriminant and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N_E}$. By $f_E = f_{E,\Delta,r} = \mathcal{I}^{\text{BF}}_{\Delta,r}(\tau, \mathcal{W}_E(z))$ we denote the twisted Bruinier-Funke theta lift of $\mathcal{W}_E(z)$ as in Section 4.1. We then have the following Hecke-equivariant diagram by Theorem 4.2.7

$$\begin{tikzcd} \mathcal{W}_E \arrow[r, "\xi_0"] \arrow[d, "\mathcal{I}^{\text{BF}}"] & G_E \arrow[d, "\text{Shintani}"] \\ f_E \arrow[r, "\xi_{1/2}"] & \mathbb{R}g_E, \end{tikzcd}$$

where g_E is the newform that maps to G_E under the Shimura correspondence.

In this section we prove the following theorem.

Theorem 6.3.1. *Assume that $E/\mathbb{Q}$ is an elliptic curve of square-free conductor N_E, and suppose that $G_E|_2 W_{N_E} = \epsilon G_E$. Denote the coefficients of $f_E(\tau)$ by $c_E^{\pm}(n,h)$. Then the following are true:*

(i) If $d \neq 1$ is a fundamental discriminant and $r' \in \mathbb{Z}$ such that $d \equiv \operatorname{sgn}(\Delta)Q(r')(\mathbb{Z})$, and $\epsilon d < 0$, then

$$L(E_d, 1) = 8\pi^2||G_E||^2||g_E||^2\sqrt{\frac{|d|}{N_E}} \cdot c_E^-(\epsilon d, r')^2.$$

(ii) *If* $d \neq 1$ *is a fundamental discriminant and* $r' \in \mathbb{Z}$ *such that* $d \equiv \operatorname{sgn}(\Delta)Q(r')(\mathbb{Z})$ *and* $\epsilon d > 0$, *then*

$$L'(E_d, 1) = 0 \quad \Longleftrightarrow \quad c_E^+(\epsilon d, r') \in \overline{\mathbb{Q}} \quad \Longleftrightarrow \quad c_E^+(\epsilon d, r') \in \mathbb{Q}.$$

Remark 6.3.2. This theorem can be seen as a more intrinsic version of Bruinier's and Ono's main theorem in [BO10]. Note that our method of proof (for part (*ii*)) is independent of the results of Bruinier and Ono. The proof also differs from the one given in [AGOR]. There, we showed that we can construct a harmonic Maass form corresponding to an elliptic curve that satisfies the hypotheses in Bruinier's and Ono's theorem.

The first part of Theorem 6.3.1 follows from the inspection of the Fourier expansion of $\xi_{1/2}(f_E)$ and Kohnen's theory of half-integral weight newforms (see Corollary 1 on page 242 of [Koh85]).

The second part is harder to prove. We will use the fact that the coefficients of f_E are given as twisted traces of CM values of $\mathcal{W}_E$, i.e. the evaluation $\mathcal{W}_E[Z]$ of $\mathcal{W}_E$ at a certain twisted Heegner divisor Z (depending on Δ and d). We relate this quantity to the coefficients of a certain differential of the third kind associated to Z. Using results of Scholl on the algebraicity of such differentials and introducing the action of the Hecke algebra we find that $\mathcal{W}_E[Z]$ is algebraic if and only if the image of the projection of Z to the G-isotypical component vanishes in the Jacobian. The Gross–Zagier Theorem then establishes the connection to the vanishing of the twisted L-derivative.

In the last part of the section we indicate implications for the algebraicity of periods of differentials of the first and second kind.

Remark 6.3.3. Note that we can also prove an analogue of part (*ii*) of Theorem 6.3.1 using the Kudla-Millson lift of $\mathcal{W}_E$ and Theorem 3.3.1. Our proof only exploits the relation between the algebraicity of the evaluation of $\mathcal{W}_E$ at the Heegner divisor and the vanishing of the projection to the G_E-isotypical component of this Heegner divisor in the Jacobian.

The similarity of the arithmetic information that the coefficients of $\mathcal{I}^{\mathrm{KM}}(\tau, \mathcal{W}_E)$ and $\mathcal{I}^{\mathrm{BF}}(\tau, \mathcal{W}_E)$ encode is also explained by the duality results in Section 5.

6.3.1. A relation between differentials of the first, second and third kind

In this section we derive a relation between differentials of the first, second and third kind defined on a compact Riemann surface X. We then specialize this result to the compactified modular curve $X_0(N)$.

We let X be a compact Riemann surface. Moreover, let $D = \sum c_i P_i$ be a degree 0 divisor on X. Let η_D be the canonical differential of the third kind with residue divisor D. From now on, we relax the condition on the integrality of the residues of η_D. We write $\partial h_D = \eta_D$ for a harmonic function h_D as in Proposition 6.1.10 and Remark 6.1.11.

Let $x_0 \in X$ be a point on X that is not contained in D. We choose a differential of the first kind ω and a differential of the second kind ϕ that only has poles at x_0 such that there is a function $F_{x_0} \in C^\infty(X)$ that satisfies $dF_{x_0} = \phi - \overline{\omega}$ (see [Can14] for a rigorous approach to this construction).

We write $F_{x_0}[D]$ for $\sum_{i=1}^n c_i F x_0(P_i)$.

Theorem 6.3.4. *Assume the notation above. Then we have*

$$F_{x_0}[D] = -\frac{1}{\pi i} \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x_0)} F_{x_0} \eta_D.$$

Proof. We consider the pairing of F_{x_0} and η_D defined by

$$[F_{x_0}, \eta_D] := \int_X (\overline{\partial} F_{x_0}) \cdot \eta_D.$$

Let $\epsilon > 0$. We can write

$$[F_{x_0}, \eta_D] = \lim_{\epsilon \to 0} \int_{X'_\epsilon} (\overline{\partial} F_{x_0})(\partial h_D),$$

with h_D as in Proposition 6.1.10 and $X'_\epsilon = X \setminus (B_\epsilon(x_0) \cup \bigcup_{i=1}^n B_\epsilon(P_i))$. We have that

$$d((\overline{\partial} F_{x_0}) \cdot h_D) = (\overline{\partial} F_{x_0})(\partial h_D) \quad \text{and} \quad d(F_{x_0}(\partial h_D)) = (\overline{\partial} F_{x_0})(\partial h_D).$$

Therefore, Stoke's Theorem implies that

$$[F_{x_0}, \eta_D] = \lim_{\epsilon \to 0} \int_{\partial X'_\epsilon} (\overline{\partial} F_{x_0}) \cdot h_D \tag{6.3.1}$$

and, at the same time,

$$[F_{x_0}, \eta_D] = \lim_{\epsilon \to 0} \int_{\partial X'_\epsilon} F_{x_0}(\partial h_D). \tag{6.3.2}$$

We first show that the integral in (6.3.1) vanishes. We compute the integrals

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(P_i)} (\overline{\partial} F_{x_0}) h_D \tag{6.3.3}$$

and

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x_0)} (\overline{\partial} F_{x_0}) h_D \tag{6.3.4}$$

separately. By Remark 6.1.11 we can write $h_D(z) = c_i \log|z - P_i| + H(z)$ for a smooth function H near P_i. We obtain that the integral in (6.3.3) is equal to

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(P_i)} (\overline{\partial} F_{x_0})(c_i \log|z - P_i| + H(z)).$$

Note that both, $\log|z - P_i|(\overline{\partial} F_{x_0})$ and $H(z)(\overline{\partial} F_{x_0})$ are continuously differentiable and thus bounded on $\partial B_\epsilon(P_i)$. Therefore, the integral vanishes as ϵ approaches 0.

We have $\overline{\partial} F_{x_0} = -\overline{\omega}$, with ω a holomorphic 1-form. Therefore, $\overline{\partial} F_{x_0}$ does not have a pole in x_0. Neither has G_D, so the integral in (6.3.4) vanishes. Thus, the integral in Equation (6.3.1) is equal to 0.

We now turn to the evaluation of the integral in (6.3.2). Note that

$$\lim_{\epsilon\to 0}\int_{\partial X'_\epsilon} F_{x_0}(\partial h_D) = -\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(x_0)} F_{x_0}(\partial h_D) - \sum_{i=1}^{n}\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(P_i)} F_{x_0}(\partial h_D). \tag{6.3.5}$$

We consider

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(P_i)} F_{x_0}\partial h_D = \lim_{\epsilon\to 0}\int_{\partial B_\epsilon(P_i)} F_{x_0}\partial\left(c_i\log|z-P_i| + H(z)\right).$$

Let U be a neighborhood of 0 in $\mathbb{C}$ and consider a chart

$$\begin{aligned} U &\xrightarrow{\phi} X \\ 0 &\mapsto P_i. \end{aligned}$$

Then we have

$$\partial\phi^*(h_D) = \phi^*(\partial h_D) = \frac{c_i}{2}\frac{dz}{z} + \partial H(z),$$

Therefore,

$$\begin{aligned} &\int_{\partial B_\epsilon(P_i)} F_{x_0}\partial\left(c_i\log|z-P_i| + H(z)\right) \\ &\qquad = \int_{\partial B_\epsilon(0)} \phi^*(F_{x_0})\phi^*(\partial\left(c_i\log|z-P_i|\right) + \int_{\partial B_\epsilon(0)} \phi^*(F_{x_0}H(z)). \end{aligned}$$

The second integral again vanishes by similar arguments as before. For the first integral we find

$$\frac{c_i}{2}\int_{\partial B_\epsilon(0)} \phi^*(F_{x_0})\frac{1}{z}dz. \tag{6.3.6}$$

Changing to polar coordinates we have that (6.3.6) equals

$$\frac{c_i}{2} i\int_0^{2\pi} \phi^*(F_{x_0}(\epsilon e^{i\theta}))d\theta.$$

Thus, the integral is independent of ϵ and as $\epsilon\to 0$ we have

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(P_i)} F_{x_0}\partial h_D = \pi i c_i F_{x_0}(P_i).$$

Summarizing, we find that

$$0 = -\pi i c_i F_{x_0}(P_i) - \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x_0)} F_{x_0} \partial h_D.$$

□

We now specify Theorem 6.3.4 to the situation when $X = X_0(N)$ is the compactified modular curve and $F \in H_0^+(N)$ is a harmonic Maass form of weight 0 for $\Gamma_0(N)$ whose principal parts at all cusps other than ∞ are constant. By Lemma 2.3.14 and Theorem 2.3.13 dF is equal to the sum of a differential of the second kind whose poles are only supported at ∞ and the complex conjugate of a differential of the first kind. We write

$$F(z) = \sum_{n \gg -\infty} a_F^+(n) q^n + \sum_{n<0} a_F^-(n) e^{2\pi i n \bar{z}}.$$

for the Fourier expansion of F.

We let D be a degree 0 divisor on $X_0(N)$ that is coprime to the cusps of $X_0(N)$. We let η_D be the associated canonical differential of the third kind with residue divisor D. We write $\eta_D = 2\pi i \sum_{n=1}^{\infty} d(n) q^n dz$ with a meromorphic modular form $\sum_{n=1}^{\infty} d(n) q^n$ of weight 2 as in Equation (6.1.5).

Corollary 6.3.5. *Assume the notation above. We have*

$$F[D] = -2 \sum_{n \geq 1} a_F^+(-n) d(n).$$

Proof. By Theorem 6.3.4 we only have to evaluate the integral over $F\eta_D$ in a neighborhood of ∞, which is equal to

$$-\frac{1}{\pi i} \lim_{t \to \infty} \int_{x=0}^{x=1} F(x+it) \eta_D(x+it) dx.$$

We plug in the Fourier expansions of F and η_D. The integral over $F^+\eta_D$ picks out the constant coefficient, that is

$$-2 \sum_{n \geq 1} a_F^+(-n) d(n).$$

For the integral over $F^-\eta_D$ we obtain similarly

$$-2 \sum_{n \geq 1} e^{-4\pi n t} a_F^-(-n) d(n),$$

which vanishes as $t \to \infty$. □

6.3.2. Hecke eigenspaces and the restriction to isotypical components

We now describe a refined version of Corollary 6.3.5 that can be obtained by considering the action of the Hecke algebra on $X_0(N)$.

Recall that the compactified modular curve $X_0(N)$ is defined over $\mathbb{Q}$. We let J be the Jacobian of $X_0(N)$ and write $J(k)$ for its points over a number field k. Then, $J(k)$ is a finitely generated abelian group. By Abel's Theorem the Jacobian can be described as the quotient of the group $\mathrm{Div}(X_0(N),k)^0$ of degree 0 divisors on $X_0(N)$ which are rational over k modulo the subgroup of principal divisors $\mathrm{div}(f)$ for $f \in k(X)^\times$.

Let $K \subset \mathbb{C}$ be a subfield of $\mathbb{C}$ and write $k \cdot K$ for the compositum of k and K (where we fix an embedding of $k \hookrightarrow \mathbb{C}$ such that $k \cdot K \in \mathbb{C}$). We let $J(k)_K = J(k) \otimes_{\mathbb{Z}} K$ and $\mathrm{Div}(X_0(N),k)^0_K = \mathrm{Div}(X_0(N),k)^0 \otimes_{\mathbb{Z}} K$.

Following Bruinier [Bru13] we denote by $\mathcal{D}(X_0(N),k)_K$ the group of meromorphic differentials on $X_0(N)$ defined over $k \cdot K$ whose poles are all of first order and whose residue divisor belongs to $\mathrm{Div}(X_0(N),k)^0_K$. Let $\mathcal{P}(X_0(N),k)_K$ be the subgroup of differentials which are finite K-linear combinations of differentials of the form $\frac{df}{f}$ with $f \in k(X)^\times$. Then we define

$$\mathcal{CL}(X_0(N),k)_K = \mathcal{D}(X_0(N),k)_K/\mathcal{P}(X_0(N),k)_K.$$

Note that the Hecke algebra $\mathbb{T}$ of $\Gamma_0(N)$ acts on $X_0(N)$ by correspondences which are defined over $\mathbb{Q}$. This induces compatible actions on $\mathcal{CL}(X_0(N),k)$, $J(k)$ and the space of holomorphic differentials whose Fourier expansions are defined over k.

Let $G \in S_2^{\mathrm{new}}(N)$ be a newform of weight 2 for $\Gamma_0(N)$. Then G is a normalized eigenform for all Hecke operators. We write $K = K_G$ for the number field generated by the Hecke eigenvalues of G. By Proposition 2.2 of [Bru13] the G-isotypical component of $J(k)_{\mathbb{C}}$ corresponding to G is defined over K.

Moreover, if $D \in \mathrm{Div}(X_0(N),k)^0_K$ is a divisor in the G-isotypical component, there is a canonical differential of the third kind $\eta_D \in \mathcal{D}(X_0(N),k)_K$ with residue divisor D whose class belongs to the G-isotypical component of $\mathcal{CL}(X_0(N),k)_K$ [Bru13, Proposition 2.2]. We let $L = k \cdot K$.

Lemma 6.3.6. *Let $G \in S_2^{\mathrm{new}}(N)$ be a newform of weight 2 for $\Gamma_0(N)$ and D be a divisor in the G-isotypical component of $\mathrm{Div}(X_0(N),k)^0_K$. We assume that the divisor D is coprime to ∞. Let $\eta_D \in \mathcal{D}(X_0(N),k)_K$ be the canonical differential of the third kind with residue divisor D whose class belongs to the G-isotypical component of $\mathcal{CL}(X_0(N),k)_K$. Denote the n-th Fourier coefficient of η_D by $d(n)$. Then we have*

$$d(n) = \lambda_n d(1) + a_n,$$

where λ_n is the eigenvalue of $T(n)$ corresponding to G and $a_i \in L$.

Proof. Let $T(n)$ be a Hecke operator and write λ_n for the eigenvalue of G under the action of $T(n)$. Since the class of η_D be belongs to the G-isotypical component we find that

$$T(n)\eta_D - \lambda_n \eta_D \in \mathcal{P}(X_0(N),k)_K.$$

Note that we can identify the differential in $\mathcal{P}(X_0(N),k)_K$ with a meromorphic differential whose Fourier coefficients are contained in L. Analyzing the action of the Hecke operator on the first Fourier coefficient of η_D, we find that

$$d(n) - \lambda_n d(1) = a_n,$$

with $a_n \in L$. □

Lemma 6.3.7. *Let the hypothesis be as in Lemma 6.3.6. Then the following are equivalent:*

(i) The first Fourier coefficient $d(1)$ of η_D is contained in L.

(ii) Some non-zero multiple of D is the divisor of a rational function.

Proof. If some non-zero multiple of D is the divisor of a rational function Corollary 6.1.15 implies that all the Fourier coefficients of η_D are contained in L. In particular, $d(1)$ is contained in L.

Now let $d(1) \in L$. We can apply the same strategy as Bruinier and Ono in the proof of Theorem 7.6 in [BO10]: Assume that D is not a principal divisor. By Corollary 6.1.15 there is a positive integer n such that $d(n)$ is transcendental. Let n_0 be the smallest of these integers. We have to show that $n_0 = 1$.

Assume that $n_0 \neq 1$ and p is a prime dividing n_0. By λ_p we denote the eigenvalue of the Hecke operator $T(p)$ corresponding to G. Then Lemma 6.3.6 implies that there is an $a_{n_0} \in L$ such that

$$d(n_0) = \lambda_p d\left(\frac{n_0}{p}\right) - pd\left(\frac{n_0}{p^2}\right) + a_{n_0},$$

with $a_{n_0} \in L$. Since $n_0/p, n_0/p^2 \leq n_0$, and n_0 was the smallest integer with the property that $d(n_0)$ is transcendental, $d(n_0)$ is a linear combination of algebraic numbers, contradicting our assumption. □

We let $H_0^{+,\infty}(N)_K$ be the space of harmonic Maass forms of weight 0 for $\Gamma_0(N)$ whose principal parts at all cusps other than ∞ vanish and whose coefficients of the principal part at ∞ are in K.

Recall that the pairing of F with a cusp form G as in Section 2.3.2 equals (in particular see the formula for the pairing in Equation (2.3.9))

$$\{G, F\} = \sum_{n\geq 1} a_F^+(n) a_G(n),$$

where we denote the n-th Fourier coefficient of G by $a_G(n)$.

Theorem 6.3.8. *Let $G \in S_2^{\mathrm{new}}(N)$ be a newform of weight 2 for $\Gamma_0(N)$ and D be a divisor in the G-isotypical component of $\mathrm{Div}(X_0(N),k)_K^0$. We assume that the divisor D is coprime to ∞. Let $F \in H_0^{+,\infty}(N)_K$ be a harmonic Maass form such that $\{G,F\} = 1$. Then the following are equivalent:*

(i) *$F\,[D]$ is contained in L.*

(ii) *Some non-zero multiple of D is the divisor of a rational function.*

Proof. Let $\eta_D \in \mathcal{D}(X_0(N), k)_K$ be the canonical differential of the third kind with residue divisor D whose class belongs to the G-isotypical component of $\mathcal{CL}(X_0(N), k)_K$.

If some non-zero multiple of D is the divisor of a rational function Theorem 6.1.14 implies that η_D is defined over L and by Corollary 6.1.15 the Fourier coefficients $d(n)$ of η_D are contained in L for all n. Recall that the coefficients of the principal part of F are contained in K. Together with Corollary 6.3.5 it follows that $F\,[D]$ is contained in L.

If $F\,[D]$ is contained in L, then by Corollary 6.3.5 we also have $\sum_{n\geq 1} a_F^+(-n)d(n) \in L$. Let $T(n)$ be a Hecke operator in $\mathbb{T}$. We write λ_n for the eigenvalue of $T(n)$ corresponding to G. Since the class of η_D belongs to the G-isotypical component Lemma 6.3.6 implies that

$$d(n) - \lambda_n d(1) = a_n,$$

with $a_n \in L$. Therefore (we omit -2 since this is not important for the algebraicity results we are looking for),

$$\begin{aligned} F[D] &= \sum_{n\geq 1} a_F^+(-n)d(n) \\ &= d(1)\sum_{n\geq 1} a_F^+(-n)a_G(n) + \sum_{n\geq 1} a_n a_F^+(-n), \end{aligned}$$

since $\lambda_n = a_G(n)$. Using that $\{G, F\} = 1$ we find

$$d(1)\sum_{n\geq 1} a_F^+(-n)a_G(n) + \sum_{n\geq 1} a_n a_F^+(-n) = d(1) + \sum_{n\geq 1} a_n a_F^+(-n).$$

By assumption we have that $F[D]$ and $\sum_{n\geq 1} a_n a_F^+(-n)$ are in L, thus $d(1) \in L$.

Lemma 6.3.7 now implies the statement in the theorem. □

Now we let $\widetilde{Z}_{\Delta,r}(m,h) = Z^+_{\Delta,r}(m,h) - Z^-_{\Delta,r}(m,h)$ be the twisted Heegner divisor defined in Section 2.7. Here, $\Delta \neq 1$ is a fundamental discriminant and $r \in \mathbb{Z}$ is such that $r^2 \equiv \Delta \pmod{4N}$. Moreover, $h \in L'/L$ and $m \in \mathbb{Q}_{>0}$ with $m \equiv \operatorname{sgn}(\Delta)Q(h)\,(\mathbb{Z})$. Recall that the divisor $\widetilde{Z}_{\Delta,r}(m,h)$ is defined over $\mathbb{Q}(\sqrt{\Delta}, \sqrt{m})$.

Corollary 6.3.9. *Let $G \in S_2^{\text{new}}(N)$ such that $K = \mathbb{Q}$ and let $G = G_1, G_2, \ldots, G_n$ be a basis of simultaneous eigenforms for $S_2(N)$. We let $F \in H_0^{+,\infty}(N)_{\mathbb{Q}}$ with the property that $\xi_0(F) = G/||G||^2$.*

Then the following are equivalent:

(i)
$$F\left[\widetilde{Z}_{\Delta,r}(m,h)\right] = \sum_{z\in \widetilde{Z}_{\Delta,r}(m,h)} F(z) \in \mathbb{Q}(\sqrt{\Delta}, \sqrt{m}).$$

(ii)

$$\frac{\sqrt{\Delta}}{\sqrt{m}} F\left[\widetilde{Z}_{\Delta,r}(m,h)\right] \in \mathbb{Q}.$$

(iii) The projection $\widetilde{Z}^{G}_{\Delta,r}(m,h)$ of the divisor $\widetilde{Z}_{\Delta,r}(m,h)$ to the G-isotypical component of $\mathrm{Div}(X_0(N),\mathbb{Q}(\sqrt{\Delta},\sqrt{m}))^0_{\mathbb{Q}}$ is a non-zero multiple of the divisor of a rational function.

Proof. We first show that (i) implies (iii). Let $F[\widetilde{Z}_{\Delta,r}(m,h)] \in \mathbb{Q}(\sqrt{\Delta},\sqrt{m})$. By Theorem 6.3.8 some non-zero multiple of $\widetilde{Z}_{\Delta,r}(m,h)$ is a principal divisor. Since the group of degree 0 divisors $\mathrm{Div}(X_0(N),\mathbb{Q}(\sqrt{\Delta},\sqrt{m}))^0_{\mathbb{Q}}$ decomposes into G-isotypical components corresponding to the basis G_1, G_2, $\dots$, G_n, it follows that a non-zero multiple of $\widetilde{Z}^{G}_{\Delta,r}(m,h)$ is a principal divisor.

Now we show that (iii) implies (i). Recall that $\xi_0(F) = G/||G||^2$ for a newform G implies that $\{G,F\} = 1$ and $\{G',F\} = 0$ for all G' orthogonal to G (see Lemma 2.3.29). Assume that the projection of $\widetilde{Z}_{\Delta,r}(m,h)$ to the G-isotypical component is a non-zero multiple of the divisor of a rational function. We write

$$\widetilde{Z}_{\Delta,r}(m,h) = \sum_{i=1}^{n} \widetilde{Z}^{G_i}_{\Delta,r}(m,h).$$

By Theorem 6.3.8 $F[\widetilde{Z}^{G}_{\Delta,r}(m,h)]$ is contained in $\mathbb{Q}(\sqrt{\Delta},\sqrt{m})$.

We now compute $F[\widetilde{Z}^{G_i}_{\Delta,r}(m,h)]$ for $i \neq 1$. By Corollary 6.3.5 we have (again omitting the factor -2)

$$F[\widetilde{Z}^{G_i}_{\Delta,r}(m,h)] = \sum_{n\geq 2} a^{+}_{F}(-n) d_i(n),$$

where $d_i(n)$ is the coefficient of the corresponding differential of the third kind whose class is in the G_i-isotypical component of $\mathcal{CL}(X_0(N),\mathbb{Q}(\sqrt{\Delta},\sqrt{m}))_{\mathbb{Q}}$. Let $T(n)$ be a Hecke operator and $\lambda_{n,i}$ be the eigenvalue of $T(n)$ corresponding to G_i. By Lemma 6.3.6 we have

$$d_i(n) - \lambda_{n,i} d_i(1) = a_{n,i},$$

with $a_{n,i} \in \mathbb{Q}(\sqrt{\Delta},\sqrt{m})$. Therefore, we find that

$$\begin{aligned} F[\widetilde{Z}^{G_i}_{\Delta,r}(m,h)] &= \sum_{n\geq 2} a^{+}_{F}(-n) d_i(n) \\ &= \sum_{n\geq 2} a^{+}_{F}(-n)\lambda_{n,i} + \sum_{n\geq 2} a^{+}_{F}(-n) a_{n,i} \\ &= \sum_{n\geq 2} a^{+}_{F}(-n) a_{G_i}(n) + \sum_{n\geq 2} a^{+}_{F}(-n) a_{n,i}. \end{aligned}$$

since $\{G_i, F\} = 0$, we have $\sum_{n\geq 2} a^{+}_{F}(-n) a_{G_i}(n) = 0$. The quantity $\sum_{n\geq 2} a^{+}_{F}(-n) a_{n,i}$ is contained in $\mathbb{Q}(\sqrt{\Delta},\sqrt{m})$ since $F \in H^{+,\infty}_0(N)_{\mathbb{Q}}$.

Obviously, (ii) implies (i). Assume that $F[\widetilde{Z}_{\Delta,r}(m,h)] \in \mathbb{Q}(\sqrt{\Delta},\sqrt{m})$. Then, we also have $\sqrt{\Delta}\sqrt{m}F[\widetilde{Z}_{\Delta,r}(m,h)] \in \mathbb{Q}(\sqrt{\Delta},\sqrt{m})$.

By Lemma 5.1 of [BO10] we have for the non-trivial automorphism σ of $\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}$ (and similarly $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$) that $\sigma(\widetilde{Z}_{\Delta,r}(m,h)) = -\widetilde{Z}_{\Delta,r}(m,h)$. Therefore,

$$\sigma\left(F\left[\widetilde{Z}_{\Delta,r}(m,h)\right]\right) = -F\left[\widetilde{Z}_{\Delta,r}(m,h)\right]$$

which then implies the desired result. □

Proof of part (ii) *of Theorem 6.3.1.* Assume that $E/\mathbb{Q}$ is an elliptic curve over $\mathbb{Q}$ of conductor N_E, and suppose that $G_E|_2W_{N_E} = \epsilon G_E$. Then the Hecke L-series of G_E satisfies a functional equation under $s \mapsto 2-s$ with root number $\epsilon_G = -\epsilon$. Note that for a fundamental discriminant D that is equal to a square modulo $4N_E$ the sign of the functional equation of $L(G,D,s)$ is $\operatorname{sgn}(D)\epsilon_G$.

Recall that $\Delta \neq 1$ is a fundamental discriminant and $r \in \mathbb{Z}$ such that $r^2 \equiv \Delta \pmod{4N}$. Moreover, we let $d \neq 1$ be a fundamental discriminant and $r' \in \mathbb{Z}$ such that $d \equiv \operatorname{sgn}(\Delta)Q(r')(\mathbb{Z})$.

We first consider the case that $d\epsilon_G < 0$ (that is $d\epsilon > 0$). Then the L-series $L(G,d,1)$ vanishes.

Recall that

$$\mathcal{W}_E(z) = \zeta(\Lambda_E;\mathcal{E}_E(z)) - S(\Lambda_E)\mathcal{E}_E(z) - M_E(z) - \frac{\deg(\phi_E)}{4\pi||G_E||^2}\overline{\mathcal{E}_E(z)},$$

where M_E was chosen such that the principal parts of $\mathcal{W}_E$ vanish at all cusps other than ∞. Moreover, $\mathcal{W}_E(z)$ was normalized such that $\xi_0(\mathcal{W}_E(z)) = G_E/||G_E||^2$. The coefficients of the principal part of $\mathcal{W}_E(z)$ at ∞ are contained in $\mathbb{Q}$ by construction. Note that the freedom in the choice of M_E does not influence our results.

For the $(\epsilon d, r)$-th coefficient of $f_E = f_{E,\Delta,r} = \mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,\mathcal{W}_E(z))$ we find

$$\begin{aligned} c_E(\epsilon d, r) &= \frac{\sqrt{|\Delta|}}{2\sqrt{|d|}}\left(\mathbf{t}^+_{\Delta,r}(\mathcal{W}_E(z);\epsilon d, r') - \mathbf{t}^-_{\Delta,r}(\mathcal{W}_E(z);\epsilon d, r')\right) \\ &= \frac{\sqrt{|\Delta|}}{2\sqrt{|d|}}\mathcal{W}_E(z)[\widetilde{Z}_{\Delta,r}(\epsilon d, r')]. \end{aligned}$$

By Corollary 6.3.9 the coefficient $c_E(\epsilon d, r)$ is rational if and only if the projection of $\widetilde{Z}_{\Delta,r}(d,r')$ to the G_E-isotypical component of $\operatorname{Div}(X_0(N),\mathbb{Q}(\sqrt{\Delta},\sqrt{d}))^0_{\mathbb{Q}}$ is a non-zero multiple of the divisor of a rational function.

By the Gross–Zagier formula (see Theorem 6.3 of [GZ86]) the Néron-Tate height on $J(H)$ of $\widetilde{Z}^{G_E}_{\Delta,r}(d,r')$ is given by

$$\langle \widetilde{Z}^{G_E}_{\Delta,r}(d,r'), \widetilde{Z}^{G_E}_{\Delta,r}(d,r')\rangle = \frac{h_K u^2}{8\pi^2||G_E||^2}\sqrt{|d\Delta|}L(G_E,\Delta,1)\cdot L'(G_E,d,1), \tag{6.3.7}$$

where H is the Hilbert class field of $K = \mathbb{Q}(\sqrt{d\Delta})$, and $2u$ is the number of roots of unity in K and h_K denotes the class number of K.

Consequently, the class of the divisor $\widetilde{Z}^{G_E}_{\Delta,r}(d,r')$ vanishes in the Jacobian if and only if we have $L(G_E,\Delta,1) = 0$ or $L'(G,d,1) = 0$. By Proposition 4.2.8 the vanishing of $L(G_E,\Delta,1)$ is equivalent to $\mathcal{W}_E(z)$ being weakly holomorphic, which is obviously not the case. This completes the proof of Theorem 6.3.1. $\square$

Remark 6.3.10. Note that the same method of proof works using the Kudla-Millson lift of $\mathcal{W}_E$. Moreover, the proof can be generalized to harmonic Maass forms $F \in H_0^{+,\infty}(N)$ mapping to a newform $G \in S_2^{\mathrm{new}}(N)$ in the straightforward way.

6.3.3. Periods of differentials of the first and second kind

In this section we explain how the results of the previous section imply conditions on the transcendence of periods of differentials of the first and second kind.

Recall that we have for a C^∞-function F by Lemma 2.3.14

$$dF = -\frac{1}{2i}\overline{\xi_0(F)}d\bar{z} + 2\pi i D(F)dz.$$

Then $\xi_0(F)dz \in S_2(N)$ is a differential of the first kind and $D(F)dz \in M_2^!(N)$ is a differential of the second kind (see (6.1.3) and (6.1.4)). We then have

$$F(z) = \left(-\frac{1}{2i}\int_p^z \overline{\xi_0(F)}d\bar{z} + 2\pi i\int_p^z D(F)dz\right),$$

for an arbitrary basepoint p.

Using the results in the previous section we directly obtain the following corollary.

Corollary 6.3.11. *Assume the notation of part (ii) of Theorem 6.3.1. Then the following are equivalent:*

(i) Some non-zero multiple of $\widetilde{Z}^{G_E}_{\Delta,r}(d,r')$ is the divisor of a rational function.

(ii) The sum of periods of differentials of the first and second kind

$$\sum_{z_{\Delta,r}\in\widetilde{Z}_{\Delta,r}(d,r')}\left(-\frac{1}{2i}\int_p^{z_{\Delta,r}}\overline{\xi_0(\mathcal{W}_E)}d\bar{z} + 2\pi i\int_p^{z_{\Delta,r}} D(\mathcal{W}_E)dz\right)$$

is rational.

Remark 6.3.12. For the Weierstrass harmonic Maass form $\mathcal{W}_E$ as in Theorem 6.3.1 we find that

$$-\frac{1}{2i}\int_p^{z_{\Delta,r}}\overline{\xi_0(\mathcal{W}_E)}d\bar{z} = -\frac{1}{2i||G_E||^2}\int_p^{z_{\Delta,r}}\overline{G_E(z)}d\bar{z}$$

and

$$2\pi i \int_p^{z_{\Delta,r}} D(\mathcal{W}_E)dz = -2\pi i \int_p^{z_{\Delta,r}} \left(\wp(\Lambda_E, \mathcal{E}_E(z))G_E(z) + S(\Lambda_E)G_E(z) + D(M_E(z))\right) dz.$$

Note that the differentials of the first and second kind we obtain above correspond to the differentials of the first and second kind on the elliptic curve E as in Example 6.1.16.

Remark 6.3.13. It would be interesting to relate these results to an expression Bruinier [Bru13] obtained for the coefficients $c_E^+(d\epsilon, r)$ in terms of periods of a certain differential of the third kind associated with the Heegner divisor $Z_{\Delta,r}(f_E)$.

6.4. An example - the elliptic curve $37a1$

We consider the elliptic curve $37a$ given by the equation

$$E : y^2 = 4x^3 - 4x + 1.$$

The sign of the functional equation is -1 and $E(\mathbb{Q})$ has rank 1.

The q-expansion of $G_E \in S_2^{\text{new}}(37)$ is given by

$$G_E(z) = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + \cdots \in S_2^{\text{new}}\left(\Gamma_0(37)\right).$$

Using Remark 6.2.4 and `Sage` [S+14] we find

$$\mathcal{W}_E^+(z) = q^{-1} + 1 + 2.1132...q + 2.3867...q^2 + 4.2201...q^3 + 5.5566...q^4 + 8.3547...q^5 + O(q^6).$$

It turns out that

$$f_E(z) = \mathcal{I}_{-3}^{\text{BF}}(\tau, \mathcal{W}(z))$$

corresponds to the Poincaré series P_{-3} with principal part $q^{-3/148}(\mathfrak{e}_{21} + \mathfrak{e}_{-21})$ (where we normalized the lift by dividing by $\sqrt{3}$).

Using `Sage` [S+14] Bruinier and Strömberg [BS12] computed the coefficients $c_{P_{-3}}^+(d)$ for fundamental discriminants $d \leq 15000$ of P_{-3} and compared them with the corresponding values of $L'(G_E, \Delta, 1) = L'(E_\Delta, 1)$.

Stephan Ehlen numerically confirmed that

$$c_E^+(d) = \frac{1}{2\sqrt{d}}\left(\mathbf{t}_{-3}^+(\mathcal{W}_E(z); d) - \mathbf{t}_{-3}^-(\mathcal{W}_E(z); d)\right)$$

using `Sage` [S+14].

Note that in the case $d = 1$ one has to pay attention to a contribution to the trace coming from the constant term of $\mathcal{W}_E$. Also note that the $\sqrt{3}$ does not appear in the denominator, since we normalized f_E to have principal part in $\mathbb{Q}$. The following table illustrates Theorem 6.3.1. It was computed by Strömberg.

d	$c_E^+(d)$	$L'(E_d, 1)$	$\mathrm{rank}(E_d(\mathbb{Q}))$
1	$-0.2817617849\ldots$	$0.3059997738\ldots$	1
12	$-0.4885272382\ldots$	$4.2986147986\ldots$	1
21	$-0.1727392572\ldots$	$9.0023868003\ldots$	1
28	$-0.6781939953\ldots$	$4.3272602496\ldots$	1
33	$0.5663023201\ldots$	$3.6219567911\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1489	9	0	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4393	66	0	3

Remark 6.4.1. In general, the task of computing the weight 1/2 harmonic Maass forms appearing in the main theorem of Bruinier and Ono [BO10, Theorem 7.8] has been non-trivial. Natural difficulties arise (see [BS12]). These weight 1/2 forms are preimages under $\xi_{1/2}$ of certain weight 3/2 cusp forms, and as mentioned earlier, there are infinitely many such preimages.

Using the methods of this thesis gives an alternative approach for the computation of the holomorphic part of a canonical harmonic Maass form.

7. Applications and examples

7.1. Powers of the Dedekind η-function

In the spirit of Bruinier and Ono who proved algebraic formulas for the coefficients of the inverse of the Dedekind η-function in terms of traces of a certain Poincaré series using the Kudla-Millson lift in [BO13], we consider $\eta(\tau)^{-25}$ here.

We let χ_{12} be the Kronecker character $\left(\frac{12}{\cdot}\right)$ and define

$$G_{25}(\tau) := \sum_{r\in\mathbb{Z}/12\mathbb{Z}} \chi_{12}(r)\eta(\tau)^{-25}\mathfrak{e}_r.$$

Using the transformation properties of the Dedekind η-function one easily sees that G_{25} is a weakly holomorphic modular form of weight $-25/2$ for the representation ρ. We prove a formula for the coefficients of $G_{25}(\tau)$ in terms of traces of weight -26 Poincaré series using Theorem 3.3.1.

We define

$$\begin{aligned} F = &-F_5(\cdot,14,-26) + F_5(\cdot,14,-26)|W_2^6 \\ &+ F_5(\cdot,14,-26)|W_3^6 - F_5(\cdot,14,-26)|W_6^6, \end{aligned}$$

and

$$\begin{aligned} \tilde{F} = (25+5^{13})\big(&F_1(\cdot,14,-26) - F_1(\cdot,14,-26)|W_2^6 \\ &-F_1(\cdot,14,-26)|W_3^6 + F_1(\cdot,14,-26)|W_6^6\big), \end{aligned}$$

where the Poincaré series F_1 and F_5 are defined as in (2.5.1).

Corollary 7.1.1. *For $n>0$ the coefficient of index $\left(\frac{24n-1}{24},1\right)$ of G_{25}, and therefore the $\frac{24n-1}{24}$-th coefficient of $\eta(\tau)^{-25}$, is given by*

$$\begin{aligned} &-\frac{185725}{4429185024\pi^{13}}\left(\frac{1}{24n-1}\right)^7 \\ &\qquad\times\left(\mathbf{t}\left(F;\frac{24n-1}{24},1\right)+\mathbf{t}\left(\tilde{F};\frac{24n-1}{24},1\right)\right). \end{aligned}$$

Remark 7.1.2. This corollary can be rephrased in terms of traces of CM points associated to quadratic forms instead of lattice elements. See the example on p. 4 of [Alf14].

Proof. The principal part of $G_{25}(\tau)$ is equal to $(q^{-25/24}+25q^{-1/24})(\mathfrak{e}_1-\mathfrak{e}_5-\mathfrak{e}_7+\mathfrak{e}_{11})$.

For the lift $\frac{1}{C^{\mathrm{o}}}\mathcal{I}^{\mathrm{KM}}(\tau,F)$ of the Poincaré series F we obtain (where C^{o} is as in Theorem 3.2.4)

$$-\sum_{n|5} n^{13}\mathcal{F}_{\frac{25}{24n^2},\frac{5}{n}}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)+\sum_{n|5} n^{13}\mathcal{F}_{\frac{25}{24n^2},\frac{5}{n}}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)|W_2^6$$
$$+\sum_{n|5} n^{13}\mathcal{F}_{\frac{25}{24n^2},\frac{5}{n}}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)|W_3^6-\sum_{n|5} n^{13}\mathcal{F}_{\frac{25}{24n^2},\frac{5}{n}}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)|W_6^6.$$

This has principal part $2(q^{-25/24}-5^{13}q^{-1/24})(\mathfrak{e}_1-\mathfrak{e}_5-\mathfrak{e}_7+\mathfrak{e}_{11})$.

The lift $\frac{1}{C^{\mathrm{o}}}\mathcal{I}^{\mathrm{KM}}(\tau,\tilde{F})$ of $\tilde{F}$ is given by

$$(25+5^{13})\left(\mathcal{F}_{\frac{1}{24},1}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)-\mathcal{F}_{\frac{1}{24},1}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)|W_2^6\right.$$
$$\left.-\mathcal{F}_{\frac{1}{24},1}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)|W_3^6+\mathcal{F}_{\frac{1}{24},1}\left(\tau,\frac{29}{4},-\frac{25}{2}\right)|W_6^6\right).$$

This has principal part $2(25+5^{13})q^{-1/24}(\mathfrak{e}_1-\mathfrak{e}_5-\mathfrak{e}_7+\mathfrak{e}_{11})$. Then the sum

$$\frac{1}{2C^{\mathrm{o}}}\left(\mathcal{I}^{\mathrm{KM}}(\tau,F)+\mathcal{I}^{\mathrm{KM}}(\tau,\tilde{F})\right)$$

has principal part $(q^{-25/24}+25q^{-1/24})(\mathfrak{e}_1-\mathfrak{e}_5-\mathfrak{e}_7+\mathfrak{e}_{11})$. Thus,

$$G_{25}(\tau)=\frac{1}{2C^{\mathrm{o}}}(\mathcal{I}^{\mathrm{KM}}(\tau,F)+\mathcal{I}^{\mathrm{KM}}(\tau,\tilde{F})),$$

which implies the formula in the corollary. $\square$

Remark 7.1.3. More generally, one can deduce formulas for the coefficients of $\eta(\tau)^{-i}$, where $i\equiv 1 \pmod{24}$. Here we let

$$G_i(\tau):=\sum_{r\in\mathbb{Z}/12\mathbb{Z}}\chi_{12}(r)\eta(\tau)^{-i}\mathfrak{e}_r.$$

Then, similarly as above, one has to construct a linear combination of twisted lifts of Poincaré series whose lift has the same principal part as $G_i(\tau)$.

7.2. A formula for the coefficients of Ramanujan's $f(q)$

In this section we present another application of Theorem 3.3.1.

We consider Ramanujan's mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2 (1+q^2)^2 \cdots (1+q^n)^2} = 1 + \sum_{n=1}^{\infty} a_f(n) q^n.$$

We let $\Delta < 0$ be a fundamental discriminant with $\Delta \equiv 1 \pmod{24}$. We define

$$F(z) = -\frac{1}{40} \frac{E_4(z) + 4E_4(2z) - 9E_4(3z) - 36E_4(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2} = q^{-1} - 4 + 83q + \cdots ,$$

which is a weakly holomorphic modular form of weight 0 for $\Gamma_0(6)$.

Corollary 7.2.1. *We have*

$$a_f\left(\frac{|\Delta|+1}{24}\right) = -\frac{1}{8i\sqrt{|\Delta|}} (\mathbf{t}_{\Delta,1}(F;1,1) - \mathbf{t}_{\Delta,1}(F;1,5) + \mathbf{t}_{\Delta,1}(F;1,7) - \mathbf{t}_{\Delta,1}(F;1,11)).$$

Remark 7.2.2. These formulas were checked numerically by Stephan Ehlen using `Sage` [S+14].

Proof. Here we employ the duality results between weight 1/2 and 3/2. Note that the function $q^{-1/24}f(q)$ can be realized as the component of the holomorphic part of a vector valued harmonic Maass form $\tilde{H}$ of weight 1/2 with representation ρ [BO10, Lemma 8.1]. More precisely,

$$H = (0, h_0, h_2 - h_1, 0, -h_1 - h_2, -h_0, 0, h_0, h_1 + h_2, 0, h_1 - h_2, -h_0)^{\mathrm{t}},$$

where the holomorphic part of h_0 is $q^{-1/24}f(q)$ and the holomorphic parts of h_1 and h_2 are given by Ramanujan's mock theta function $\omega(q)$. The non-holomorphic parts are given by certain unary theta series [BO10, Section 8.2]. The principal part of H is given by $q^{-1/24}(\mathfrak{e}_1 - \mathfrak{e}_5 + \mathfrak{e}_7 - \mathfrak{e}_{11})$.

In terms of Poincaré series we have

$$F(z) \doteq F_1(z,1,0) + F_1(z,1,0)|W_2^6 - F_1(z,1,0)|W_3^6 - F_1(z,1,0)|W_6^6,$$

where $\doteq$ means up to addition of a constant.

By Theorem 3.4.1 we see that $\mathcal{I}^{\mathrm{KM}}_{\Delta,1}(\tau, F)$ is a weakly holomorphic modular form. Note that the non-holomorphic part vanishes since 6 is square-free and for $\Delta < 0$ we have $\chi_\Delta(-\lambda) = -\chi_\Delta(\lambda)$. To determine the principal part of $\mathcal{I}^{\mathrm{KM}}_{\Delta,1}(\tau, F)$ we compute the lift of the Poincaré series. Note that the lift of a constant vanishes in this case.

By Theorem 3.2.4 the function $F(z)$ lifts to a vector valued Poincaré series having principal part

$$2i\,|\Delta|^{1/2}\, q^{-|\Delta|/24}(\mathfrak{e}_1 - \mathfrak{e}_5 + \mathfrak{e}_7 - \mathfrak{e}_{11}).$$

Therefore, by Proposition 2.3.20 we obtain that $\left\{\mathcal{I}^{\mathrm{KM}}_{\Delta,1}(\tau,F),H\right\}=0$, which implies

$$\begin{aligned}
&c_H^+(-1,1)\mathbf{t}_{\Delta,1}(F;1,1)+c_H^+(-1,5)\mathbf{t}_{\Delta,1}(F;1,5)\\
&\qquad+c_H^+(-1,7)\mathbf{t}_{\Delta,1}(F;1,7)+c_H^+(-1,11)\mathbf{t}_{\Delta,1}(F;1,11)\\
&=-2i\sqrt{|\Delta|}\left(c_H^+\left(\frac{|\Delta|}{24},1\right)\cdot 1+c_H^+\left(\frac{|\Delta|}{24},5\right)\cdot(-1)\right.\\
&\qquad\left.+c_H^+\left(\frac{|\Delta|}{24},7\right)\cdot 1+c_H^+\left(\frac{|\Delta|}{24},11\right)\cdot(-1)\right).
\end{aligned}$$

Since we can identify the coefficients in the different components of H we obtain the formula in the corollary. □

Remark 7.2.3. For $\Delta\equiv r^2 \pmod{24}$, where $r\equiv 5,7,11 \pmod{12}$, we consider

$$F(z)=-\frac{1}{40}\frac{E_4(z)\pm 4E_4(2z)\pm 9E_4(3z)\pm 36E_4(6z)}{\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2}=q^{-1}-4+83q+\cdots$$

and have to arrange the $\pm$'s in such a way that we obtain (up to a constant) $q^{-|\Delta|/24}(\mathfrak{e}_1-\mathfrak{e}_5+\mathfrak{e}_7-\mathfrak{e}_{11})$ as the principal part of the lift.

Remark 7.2.4. A priori the lift of a constant is not a harmonic Maass form in the +-space. Therefore, it is not possible to obtain the same results using the Bruinier-Funke lift right away.

Remark 7.2.5. In general, given a scalar valued form one has to realize it as the component of a vector valued harmonic weak Maass form to be able to obtain a formula as above for its coefficients. For a detailed discussion of this problem see a preprint of Fredrik Strömberg [Str]. If the corresponding vector valued form is known, one can construct the input function using Poincaré series.

7.3. The example $\Gamma_0(p)$ of the introduction

We explain how to obtain the theorems in the introduction. Let p be a prime and let $\Delta>1$ be a fundamental discriminant satisfying $(\Delta,2p)=1$ and let $r\in\mathbb{Z}$ with $\Delta\equiv r^2 \pmod{4p}$. Moreover, let $F\in H^+_{-2k}(p)$ be a harmonic Maass form of negative weight $-2k$ for $\Gamma_0(p)$ that is invariant under the Fricke involution.

The group $\Gamma_0(p)$ has two cusps ∞ and 0. These two cusps are interchanged by the Fricke involution.

Via mapping $\sum_{h\in L'/L}f_h(\tau)\mathfrak{e}_h$ to $\sum_{h\in L'/L}f_h(4p\tau)$ we obtain an isomorphism between the spaces $H^+_{1/2-k,\rho}$ and $H^+_{1/2-k}(4p)$ and $M^!_{3/2+k,\widetilde{\rho}}$ and $M^!_{3/2+k}(4p)$ if k is odd. If k is even, we obtain isomorphisms between $H^+_{1/2-k,\widetilde{\rho}}$ and $H^+_{1/2-k}(4p)$ and $M^!_{3/2+k,\rho}$ and $M^!_{3/2+k}(4p)$. For $M^!_{k/2,\rho}$ this is Theorem 5.6 in [EZ85]; the isomorphism extends to $H^+_{k/2,\rho}$.

The assumption $(\Delta, 2p) = 1$ guarantees that we can choose $r \in \mathbb{Z}$ as a unit in $\mathbb{Z}/4p\mathbb{Z}$. Therefore, the sum $\sum_{h\in L'/L} \mathcal{I}^{\mathrm{KM}}_{\Delta,r,h}$ does not depend on r. Here, we wrote $\mathcal{I}^{\mathrm{KM}}_{\Delta,r,h}$ for the h-th component of the theta lifts.

Recall that $-d$ is a negative fundamental discriminant such that $-d$ and Δ are squares modulo $4p$. By $\mathcal{Q}_{-d\Delta,p}$ we denote the set of integral binary quadratic forms $[a, b, c] = ax^2+bxy+cy^2$ of discriminant $-d\Delta$ such that $c \equiv 0 \pmod p$. We assume that $(\Delta, 2p) = 1$ if $p \neq 1$.

As described in Section 1.2.1 we identify lattice elements with integral binary quadratic forms. Recall that the action of the group $\Gamma_0(p)$ on both spaces is compatible. Notice that we have to consider positive and negative definite quadratic forms. For positive Δ we have $\chi_\Delta(-Q) = \chi_\Delta(Q)$ which yields for $m = \frac{d}{4p} > 0$ that

$$\sum_{h\in L'/L} \sum_{\lambda\in\Gamma_0(p)\backslash L_{rh,|\Delta|m}} \frac{\chi_\Delta(\lambda)}{|\overline{\Gamma}_\lambda|} \partial F(D_\lambda) = \sum_{Q\in\Gamma_0(p)\backslash\mathcal{Q}_{-d|\Delta|,p}} \frac{\chi_\Delta(Q)}{|\overline{\Gamma_0(p)}_Q|} \partial F(\alpha_Q).$$

For the coefficients of the holomorphic part of $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F)$ we proceed analogously, now assuming that $(d, 2p) = 1$. (Moreover, we require that the constant coefficients of F vanish at all cusps if the weight of F is zero.)

7.4. The lift of $\log(||\Delta||)$

In this section we compute the lift of $\log(||\Delta||)$. We let

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

be the Delta function. We normalize the Petersson metric of Δ such that

$$||\Delta(z)|| = |\Delta(z)y^6|.$$

Theorem 7.4.1. *Let $N = 1$ and $\Delta < 0$. Then we have*

$$-\frac{1}{12}\mathcal{I}^{BF}_{\Delta,r}(\tau, \log(|\Delta(z)y^6|)) = \frac{1}{12}|\Delta|\Lambda(\epsilon_\Delta, 1) \cdot \left(G(\tau) - \frac{6}{\pi}\Theta_K(\tau)(\gamma - \log(4\pi))\right),$$

where $G(\tau)$ is as in Proposition A.0.5.

Remark 7.4.2. This might be interpreted as a second term identity in the sense of Kudla and Rallis.

Remark 7.4.3. The Kudla-Millson lift of $\log ||\Delta||$ was computed in [BF06] and [AE13].

Proof. Recall that

$$\zeta^*(2s) = \frac{1}{2}\frac{1}{s-\frac{1}{2}} + \frac{1}{2}(\gamma - \log(4\pi)) + O(s-1/2),$$

and

$$\mathcal{E}_0(z,s) = \frac{1}{2}\frac{1}{s-\frac{1}{2}} + \frac{1}{2}(\gamma - \log(4\pi)) - \frac{1}{12}\log(|\Delta(z)y^6|) + O(s-1/2)$$

by the Kronecker limit formula.

Therefore

$$\lim_{s\to\frac{1}{2}}(\mathcal{E}_0(z,s) - \zeta^*(2s)) = -\frac{1}{12}\log(|\Delta(z)y^6|).$$

So we have

$$-\frac{1}{12}\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \log(|\Delta(z)y^6|)) = \lim_{s\to\frac{1}{2}}\left(\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \mathcal{E}_0(z,s)) - \zeta^*(2s)\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau,1)\right).$$

We let

$$C := \frac{1}{12}|\Delta|\Lambda(\epsilon_\Delta, 1)$$

and obtain that

$$\begin{aligned}
-&\frac{1}{12}\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, \log(|\Delta(z)y^6|)) \\
&= C\cdot\lim_{s\to\frac{1}{2}}\Bigg(\left(\frac{\mathrm{res}_{s=1/2}\mathcal{E}_{1/2,K}(\tau,s)}{s-\frac{1}{2}} + \mathrm{CT}_{s=1/2}(\mathcal{E}_{1/2,K}(\tau,s)) + O(s-\frac{1}{2})\right) \\
&\quad - 2(\mathrm{res}_{s=1/2}\mathcal{E}_{1/2,K}(\tau,s))\left(\frac{1/2}{s-\frac{1}{2}} + \frac{1}{2}(\gamma - \log(4\pi)) + O(s-\frac{1}{2})\right)\Bigg) \\
&= C\cdot\left(\mathrm{CT}_{s=1/2}(\mathcal{E}_{1/2,K}(\tau,s)) - 2(\mathrm{res}_{s=1/2}\mathcal{E}_{1/2,K}(\tau,s))\frac{1}{2}(\gamma - \log(4\pi))\right).
\end{aligned}$$

By using Proposition 4.2.9 and A.0.5 we obtain the result. □

8. Outlook

There are several open problems closely related to the results in this thesis that we did not work out at the time finishing this thesis. In this chapter we briefly describe some of them.

The coefficients of the non-holomorphic part of $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F)$ It remains to compute the coefficients of the non-holomorphic part of the Bruinier-Funke lift $\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F) \in H_{1/2,\widetilde{\rho}}$ of a harmonic Maass form F of weight 0. In view of Theorem 4.2.7 these coefficients encode information in terms of F on the coefficients of the cusp form $\xi_{1/2}(\mathcal{I}^{\mathrm{BF}}_{\Delta,r}(\tau, F)) \in S_{3/2,\bar{\widetilde{\rho}}}$.

It might be possible to construct a Green current for the Millson Schwartz function similar to the ones for the Kudla-Millson and Siegel Schwartz functions as in [KM86, BF06, BFI13].

Relation to the Shintani lifting Another interesting question is if the Bruinier-Funke lift is related to the Shintani lift when the weight of the input function is negative. That is, we are looking for a relation similar to the one in Theorem 4.2.7.

Lifts of other types of automorphic forms In Theorem 4.2.10 we computed the Bruinier-Funke lift of the non-holomorphic weight 0 Eisenstein series. It would be interesting to consider the lift of other types of automorphic forms, for example $\log ||f||$ for a meromorphic modular form f (see also the work of Funke on the Kudla-Millson lifts of such forms [Fun07]).

Relation to a result on p-adic modular forms The alternative proof we gave for Bruinier's and Ono's main theorem (see the proof of Theorem 6.3.1) might help to understand the relation of this theorem to its p-adic analog of Darmon–Tornaria [DT08, Theorem 1.5].

A. The non-holomorphic Eisenstein series of weight $1/2$

Let $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\Re(s) > 1$. We compute the Fourier expansion of the weight $1/2$ non-holomorphic Eisenstein series

$$P_{0,0}(\tau, s) = \frac{1}{2} \sum_{\gamma \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \left[v^{s-1/4} \mathfrak{e}_0 \right] |_{1/2, \overline{\rho}} \gamma \,. \tag{A.0.1}$$

This Eisenstein series is a special case of the vector valued Poincaré series $P_{m,h}(\tau, s)$ considered in [BFI13]. To the best knowledge of the author the Fourier expansion of the Eisenstein series has not been computed yet. However, the Fourier expansion of similar Poincaré series has been computed by Bruinier in [Bru02] whose strategy we follow here.

We let

$$\mathcal{W}_n(v, s) = \begin{cases} |n|^{-1/4} \Gamma \left(s + \frac{\mathrm{sgn}(n)}{4} \right)^{-1} (4\pi v)^{-1/4} W_{1/4 \mathrm{sgn}(n), s-1/2}(4\pi |n| v) & \text{if } n \neq 0, \\ \frac{2^{2s-\frac{1}{2}}}{(2s-1)\Gamma(2s-1/2)} v^{3/4-s} & \text{if } n = 0, \end{cases}$$

where $W_{k,s}$ denotes the usual W-Whittaker function.

Proposition A.0.4. *We have*

$$P_{0,0}(\tau, s) = 2v^{s-1/4} \mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z} + q(\gamma)} b(n, \gamma, s) \mathcal{W}_n(v, s) e^{2\pi i n u} \mathfrak{e}_\gamma,$$

where

$$b(n, \gamma, v) = \begin{cases} (-2\pi)(4\pi |n|)^{1/4-s} (2\pi n)^{2s-1} \sum_{c \neq 0} |c|^{1-2s} H_c^*(0, 0, \gamma, n) & \textit{if } n \neq 0, \\ \sqrt{\pi} \Gamma(2s-1) 2^{1-2s} \sum_{c \neq 0} |c|^{1-2s} H_c^*(0, 0, \gamma, 0) & \textit{if } n = 0. \end{cases}$$

Here, we denote by

$$H_c^*(\beta, m, \gamma, n) = \frac{e^{-\pi i \, \mathrm{sgn}(c)/4}}{|c|} \sum_{\substack{d(c)^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) / \Gamma_\infty}} \rho_{\gamma\beta} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} e \left(\frac{ma + nd}{c} \right)$$

the generalized Kloosterman sum. The sum runs over all primitive residues d modulo c.

Moreover, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a representative for the double coset in $\Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$ with lower row (c, d') such that $d' \equiv d \pmod{c}$.

Proof. We proceed as in [Bru02, Theorem 1.9]. We split the sum in equation (A.0.1) into the sum over 1, Z, Z^2, $Z^3 \in \widetilde{\Gamma}_\infty \setminus \mathrm{Mp}_2(\mathbb{Z})$ and $(M, \phi) \in \widetilde{\Gamma}_\infty \setminus \mathrm{Mp}_2(\mathbb{Z})$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$. Since $\mathfrak{e}_0 \big|_{1/2,\overline{\rho}} Z = \mathfrak{e}_0$ we find for the first part

$$2v^{s-1/4}\mathfrak{e}_0.$$

We now compute the Fourier expansion of the latter part, which we denote by $G(\tau, s)$. Since $\mathfrak{e}_0$ is invariant under the action of Z^2, we can write $G(\tau, s)$ in the form

$$\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z}) \\ c \neq 0}} \left[v^{s-1/4}\mathfrak{e}_0\right]\big|_{1/2,\overline{\rho}} \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}.$$

We let $\gamma \in L'/L$ and $n \in \mathbb{Z}+q(\gamma)$ and we write $c(n, \gamma, v)$ for the (n, γ)-th Fourier coefficient of $G(\tau, s)$. Then we have

$$\begin{aligned} c(n, \gamma, v) &= \int_0^1 \left\langle G(\tau, s), e^{2\pi i n u}\mathfrak{e}_\gamma \right\rangle du \\ &= \sum_{\substack{c \neq 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty}} \int_{-\infty}^{\infty} (c\tau + d)^{-1/2} \left(\frac{v}{|c\tau + d|^2}\right)^{s-1/4} \left\langle \overline{\rho}^{-1}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\mathfrak{e}_0, e^{2\pi i n u}\mathfrak{e}_\gamma \right\rangle du. \end{aligned}$$

Since ρ is unitary we have

$$\left\langle \overline{\rho}^{-1}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\mathfrak{e}_0, e^{2\pi i n u}\mathfrak{e}_\gamma \right\rangle = \rho_{\gamma 0}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} e^{-2\pi i n u}.$$

Therefore,

$$c(n, \gamma, v) = \sum_{\substack{c \neq 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty}} \rho_{\gamma 0}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \int_{-\infty}^{\infty} (c\tau + d)^{-1/2} \left(\frac{v}{|c\tau + d|^2}\right)^{s-1/4} e^{-2\pi i n u} du.$$

Using $(c\tau + d)^{1/2} = \mathrm{sgn}(c)\sqrt{c}\sqrt{\tau + d/c}$ and substituting u by $u - d/c$ we obtain

$$\begin{aligned} c(n, \gamma, v) = &\sum_{\substack{c \neq 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \setminus \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty}} \rho_{\gamma 0}\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} e^{2\pi i n d/c} |c|^{-1/2} \,\mathrm{sgn}(c)^{1/2} \\ &\times \int_{-\infty}^{\infty} \tau^{-1/2} \left(\frac{v}{|c|^2|\tau|^2}\right)^{s-1/4} e^{-2\pi i n u} du. \end{aligned}$$

Using the definition of the generalized Kloosterman sum we obtain

$$\begin{aligned} c(n,\gamma,v) &= \sum_{c\neq 0} |c|^{1/2} i^{1/2} H_c^*(0,0,\gamma,n) \int_{-\infty}^{\infty} \tau^{-1/2} \left(\frac{v}{|c|^2|\tau|^2}\right)^{s-1/4} e^{-2\pi i n u} du \\ &= \sum_{c\neq 0} |c|^{1/2} H_c^*(0,0,\gamma,n) \left(\frac{v}{c^2}\right)^{-1/4} \int_{-\infty}^{\infty} \left(\frac{\tau}{-\bar\tau}\right)^{-1/4} \left(\frac{v}{c^2|\tau|^2}\right)^{s} e^{-2\pi i n u} du. \end{aligned}$$

The latter integral equals

$$\int_{-\infty}^{\infty} \left(\frac{v-iu}{v+iu}\right)^{-1/4} \left(\frac{v}{c^2|\tau|^2}\right)^{s} e^{-2\pi i n u} du \tag{A.0.2}$$

and by substituting $u = \frac{vx}{2\pi n}$ we find

$$v^{1-s} c^{-2s} (2\pi n)^{2s-1} \int_{-\infty}^{\infty} (2\pi n - ix)^{-1/4-s} (2\pi n + ix)^{1/4-s} e^{-ivx} dx.$$

This integral is a Fourier transform for $n \neq 0$, which is computed in [EMOT54, p. 119, eq. (12)]. It equals

$$-\frac{2\pi}{\Gamma\left(s+\operatorname{sgn}(n)\frac{1}{4}\right)} (4\pi|n|)^{-s} v^{s-1} W_{1/4\operatorname{sgn}(n),1/2-s}(4\pi|n|v).$$

In the case $n=0$ we find that (A.0.2) equals

$$c^{-2s} v^{1-s} \int_{-\infty}^{\infty} (1-ix)^{-1/4-s}(1+ix)^{1/4-s} dx = \begin{cases} c^{-2s} v^{1-s} \sqrt{2\pi} \frac{\Gamma(2s-1)}{\Gamma\left(2s-\frac{1}{2}\right)} & \text{if } \Re(s) > \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$c(n,\gamma,v) = \begin{cases} (-2\pi)(4\pi|n|)^{1/4-s}(2\pi n)^{2s-1} \mathcal{W}_n(v,s) \sum_{c\neq 0} |c|^{1-2s} H_c^*(0,0,\gamma,n) & \text{if } n\neq 0, \\ \sqrt{\pi}\Gamma(2s-1) 2^{1-2s} \mathcal{W}_n(v,s) \sum_{c\neq 0} |c|^{1-2s} H_c^*(0,0,\gamma,0) & \text{if } n = 0. \end{cases}$$

□

To compute the constant term of $\mathcal{E}_{1/2,\bar\rho_K}(\tau,s) = P_{0,0}\left(\tau, \frac{s}{2}+\frac{1}{2}\right)$ at $s=\frac{1}{2}$ note that

$$\mathcal{W}_n\left(v, \frac{3}{4}\right) = \mathcal{W}_n(y)$$

for $n \neq 0$, where

$$\mathcal{W}_n(v) = e^{-2\pi n v} \begin{cases} |n|^{-1/2} \beta(4\pi|n|v) & \text{if } n < 0, \\ n^{-1/2} & \text{if } n > 0. \end{cases}$$

For $n = 0$ we have $\mathcal{W}_0(v, 3/4) = 1$.

Proposition A.0.5. *The constant term of* $\mathcal{E}_{1/2,\overline{\rho}_K}(\tau, s)$ *is given by*

$$
\begin{aligned}
G(\tau) &= 2v^{1/2}\mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sqrt{2\pi}\,\mathrm{CT}_{s=\frac{1}{2}} \left(\sum_{c \neq 0} |c|^{1-2s} H_c^*(0,0,\gamma,0) \right) \mathfrak{e}_\gamma \\
&+ \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n>0}} (-2\pi)(4\pi)^{-1/2} n^{-3/4} \mathrm{CT}_{s=\frac{1}{2}} \left(\sum_{c \neq 0} |c|^{1-2s} H_c^*(0,0,\gamma,n) \right) q^n \mathfrak{e}_\gamma \\
&+ \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+q(\gamma) \\ n<0}} (-2\pi)(4\pi)^{-1/2} |n|^{-3/4} \mathrm{CT}_{s=\frac{1}{2}} \left(\sum_{c \neq 0} |c|^{1-2s} H_c^*(0,0,\gamma,n) \right) \beta(4\pi|n|v) q^n \mathfrak{e}_\gamma.
\end{aligned}
$$

Remark A.0.6. Using the strategy of Bruinier and Kuss in [BK01] and Proposition 2.2.7 we can explicitly evaluate the Kloosterman sums $H_c^*(0,0,\gamma,n)$ and the resulting L-series $\sum_{c \neq 0} |c|^{1-2s} H_c^*(0,0,\gamma,n)$.

List of Symbols

(λ, μ)	The bilinear form associated to Q, later $(\lambda, \mu) = -N\,\mathrm{tr}(\lambda \cdot \mu)$, 18
(a, b)	$= \gcd(a, b)$, the greatest common divisor of a and b
(b^+, b^-)	The signature of a quadratic space
$(f, g)_{k,\rho_L}^{\mathrm{reg}}$	The regularized Petersson inner product of f and g, 34
$(f, g)_k$	The Petersson inner product of f and g, 25
$\{f, g\}$	A bilinear pairing of f and g, 35
$[a, b, c]$	A binary quadratic form, 15
λ	An element of L
A_{k,ρ_L}	The space of functions that transform of weight k with respect to the representation ρ_L, 30
$\mathbb{C}$	The field of complex numbers
$\mathbb{C}[L'/L]$	The group algebra of a lattice L, 27
c_λ	$= \{z \in D \,:\, z \perp \lambda\}$, 21
$c_E^\pm$	The coefficients of f_E, 100
c_f^+	The coefficients of the holomorphic part of a harmonic Maass form f
c_f^-	The coefficients of the non-holomorphic part of a harmonic Maass form f
Δ	A fundamental discriminant
Δ_k	The hyperbolic weight k Laplace operator, 29
D	Usually the real hyperbolic space of dimension 2, 19
d	A fundamental discriminant
$D(f)(\tau)$	$= \frac{1}{2\pi i}\frac{\partial}{\partial\tau} f$, a differential operator, 33
$\mathcal{E}_E$	The Eichler integral of a cusp form G_E, 97

ϵ_Δ	$\epsilon_\Delta = 1$ if $\Delta > 0$ and $\epsilon_\Delta = i$ if $\Delta < 0$
$\mathcal{E}_0(z, s)$	The (normalized) real-analytic Eisenstein series of weight 0 for $\Gamma_0(N)$, 75
$\mathcal{E}_{1/2,K}(\tau, s)$	A real-analytic Eisenstein series of weight 1/2, 75
$\mathfrak{e}_h$	The standard basis elements of $\mathbb{C}[L'/L]$
$e(x)$	$= e^{2\pi i x}$
E	An elliptic curve over $\mathbb{Q}$, 90
E_Δ	The Δ-quadratic twist of an elliptic curve E, 93
$F_m(z, s, k)$	A Poincaré series, 40
$\mathcal{F}_{m,h}(\tau, s, k)$	A Poincaré series, 40
F	Usually a harmonic Maass form of weight $-2k$ for $\Gamma_0(N)$
f^+	The holomorphic part of a harmonic Maass form f, 31
f^-	The non-holomorphic part of a harmonic Maass form f, 31
f_E	The Bruinier-Funke lift of $\mathcal{W}_E$, 100
f_h	The h-th component of a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$, 29
$\Gamma(a, x)$	The incomplete Γ-function, 31
γ	Usually an element of $\Gamma_0(N)$, later also the Euler-Mascheroni constant
g	An element of $\mathrm{SL}_2(\mathbb{R})$
$G(\tau)$	The constant term of $\mathcal{E}_{1/2,K}(\tau, s)$ at $s = \frac{1}{2}$, 76
$g.\lambda$	$= g\lambda g^{-1}$, conjugation, 18
G_E	A cusp form of weight 2 for $\Gamma_0(N_E)$ associated to an elliptic curve E
G_{2k}	The weight $2k$ Eisenstein series, 91
$\Gamma_0(N)$	$= \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N}\}$
Γ_∞	$= \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$, 39
$\overline{\Gamma}_\lambda$	The stabilizer of λ in the image of $\Gamma_0(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$, 21
$\overline{\Gamma}_Q$	The stabilizer of Q in the image of $\Gamma_0(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$, 3
$\widetilde{\Gamma}_\infty$	$= \{(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), 1) : n \in \mathbb{Z}\}$, 27

η_D	A canonical differential of the third kind with residue divisor D, 95
$\mathbb{H}$	The complex upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} \ : \Im(z) > 0\}$
h	An element of L'/L
$H_k^+(N)$	The space of harmonic Maass forms of weight k for $\Gamma_0(N)$, 30
H_{k,ρ_L}^+	The space of harmonic Maass forms of weight k with respect to the representation ρ_L and the group $\mathrm{Mp}_2(\mathbb{Z})$, 29
$H_c(0,0,\gamma,n)$	A generalized Kloosterman sum, 76
$H_k(N)$	The space of harmonic Maass forms of weight k for $\Gamma_0(N)$, 30
H_{k,ρ_L}	The space of harmonic Maass forms of weight k with respect to the representation ρ_L and the group $\mathrm{Mp}_2(\mathbb{Z})$, 29
$\mathcal{I}_{\Delta,r}^{\mathrm{KM}}(\tau,F)$	The Kudla-Millson theta lift of a function F, 52
$\mathcal{I}_{\Delta,r}^{\mathrm{BF}}(\tau,F)$	The Bruinier-Funke theta lift of a function F, 68
$\mathcal{I}_{\Delta,r}^{\mathrm{Sh}}(\tau,G)$	The Shintani theta lift of a function G, 73
$\mathrm{Iso}(V)$	The set of isotropic lines in V, 20
$\Im(z)$	The imaginary part of z
$J_{k,m}$	The space of holomorphic Jacobi forms of weight k and index m, 37
$L_{\|\Delta\|m,rh}^+$	A subset of $L_{\|\Delta\|m,rh}$, 49
$L_{\|\Delta\|m,rh}^-$	A subset of $L_{\|\Delta\|m,rh}$, 49
ℓ	An element of $\mathrm{Iso}(V)$
Λ	A lattice in $\mathbb{C}$
Λ_E	A lattice in $\mathbb{C}$ corresponding to an elliptic curve E, 91
L	A lattice, later $L = \{\left(\begin{smallmatrix} b & -a/N \\ c & -b \end{smallmatrix}\right) \ : \ a,b,c \in \mathbb{Z}\}$, 18
L'	The dual lattice of L, 18
L^-	$= (L, Q^-)$, 17
L_k	The Maass lowering operator, 32
L_k^n	The iterated Maass lowering operator, 32

$L_{m,h}$	$= \{\lambda \in L + h \,:\, Q(\lambda) = m\}$, 19
$L(E_\Delta, s)$	The twisted L-function of an elliptic curve E, 93
$L(f, D, s)$	The twisted L-function of a cusp form f, 26
$L(E, s)$	The L-function of an elliptic curve, 91
$L(f, s)$	The L-function of a cusp form f, 25
$\mathcal{M}_{s,k}(y)$	$= y^{-k/2} M_{-\frac{k}{2}, s-\frac{1}{2}}(y)$, 39
$\mathrm{Mp}_2(\mathbb{R})$	The metaplectic group, 26
$\mathrm{Mp}_2(\mathbb{Z})$	The metaplectic group over the integers, 27
M	The modular curve $\Gamma_0(N) \setminus D$, 20
m	Usually $m \in \mathbb{Q}$, later $m \in \mathbb{Q}_{>0}$ such that $m \equiv \mathrm{sgn}(\Delta) Q(h) \pmod{\mathbb{Z}}$
$M_k^!(N)$	The space of weakly holomorphic modular forms of weight k for $\Gamma_0(N)$, 24
$M_{k,\rho_L}^!$	The space of weakly holomorphic modular forms of weight k with respect to the representation ρ_L and the group $\mathrm{Mp}_2(\mathbb{Z})$, 30
$M_k(N)$	The space of modular forms of weight k for $\Gamma_0(N)$, 24
M_{k,ρ_L}	The space of modular forms of weight k with respect to the representation ρ_L and the group $\mathrm{Mp}_2(\mathbb{Z})$, 30
N	A positive integer
$\mathrm{O}(-)$	The orthogonal group of $-$, 16
$\mathbb{P}^1(\mathbb{Q})$	$= \mathbb{Q} \cup \{\infty\}$, 20
$\wp(\Lambda; z)$	The Weierstrass $\wp$-function, 90
P_f	The principal part of a harmonic Maass form f, 29
Q	A quadratic form, later $Q(\lambda) = N \det(\lambda)$, 18
q	Usually $q = e^{2\pi i \tau}$ or $q = e^{2\pi i z}$
$\overline{\rho}$	The dual representation of ρ
ρ	$= \rho_L$ for the lattice $L = \{\left(\begin{smallmatrix} b & -a/N \\ c & -b \end{smallmatrix}\right) \,:\, a, b, c \in \mathbb{Z}\}$
ρ_L	The Weil representation on $\mathbb{C}[L/L']$, 28

$\widetilde{\rho}$	$=\rho$ if $\Delta > 0$ and $\overline{\rho}$ if $\Delta < 0$
r	An integer satisfying $r^2 \equiv \Delta \pmod{4N}$
r'	An integer satisfying $r'^2 \equiv d \pmod{4N}$
$R(\lambda, z)$	$= \frac{1}{2}(\lambda, \lambda(z))^2 - (\lambda, \lambda)$, 42
R_k	The Maass raising operator, 32
R_k^n	The iterated Maass raising operator, 32
$\Re(z)$	The real part of z
$\mathrm{SL}_2(\mathbb{R})$	The space of 2×2-matrices with real entries and determinant 1
$\mathrm{SL}_2(\mathbb{Z})$	The space of 2×2-matrices with integer entries and determinant 1
s	$s \in \mathbb{C}$
$S_k^{\mathrm{new}}(N)$	The space of newforms of weight k for $\Gamma_0(N)$, 25
$S_k^{\mathrm{old}}(N)$	The space of oldforms of weight k for $\Gamma_0(N)$, 25
$S_k(N)$	The space of cusp forms of weight k for $\Gamma_0(N)$, 24
S_{k,ρ_L}	The space of cusp forms of weight k with respect to the representation ρ_L and the group $\mathrm{Mp}_2(\mathbb{Z})$, 30
$\mathrm{SO}(-)$	The special orthogonal group of $-$, 16
$\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{KM}})$	The Kudla-Millson theta function, 46
$\Theta_L(\tau, z, \varphi_{\mathrm{S}})$	The Siegel theta function, 42
$\Theta_{\Delta,r}(\tau, z, \psi_{\mathrm{KM}})$	The Millson theta function, 44
$\Theta_{\Delta,r}(\tau, z, \varphi_{\mathrm{Sh}})$	The Shintani theta function, 47
$\tilde{\Theta}_{K_\ell}(\tau)$	A theta series associated to a cusp ℓ, 75
$T(p)$	A Hecke operator, 24
τ	$\tau = u + iv \in \mathbb{H}$
$\mathbf{t}^{+}_{\Delta,r}(F; m, h)$	A modular trace function, 49
$\mathbf{t}^{-}_{\Delta,r}(F; m, h)$	A modular trace function, 49
$\mathbf{t}_{\Delta,r}(F; m, h)$	A modular trace function, 49

ϕ_E	The modular parametrization of E, 92
V	The rational quadratic space of signature $(1,2)$ realized as the space of 2×2-matrices with rational entries and trace 0, 17
$\varphi_{\mathrm{KM}}(\lambda, z)$	The Kudla-Millson theta kernel, 46
$\varphi_{\mathrm{S}}(\lambda, \tau, z)$	The Siegel theta kernel, 42
$\varphi_{\mathrm{Sh},\Delta}(\lambda, \tau, z)$	The Shintani theta kernel, 47
$\mathcal{W}_{s,k}(y)$	$= y^{-k/2}W_{k/2,s-1/2}(y)$, 41
$\mathcal{W}_E$	The Weierstrass harmonic Maass form for a cusp form G_E, 98
W_N	The Fricke involution, 20
W_Q^N	An Atkin-Lehner involution, 20
$\chi_\Delta(\delta)$	A generalized genus character for $\delta \in L'$, 39
χ_D	$= \left(\frac{D}{\cdot}\right)$, the Kronecker character associated to a fundamental discriminant D, 26
$\xi_k(f)$	$= v^{k-2}\overline{L_k f}$, a differential operator, 33
X	A compact Riemann surface, 94
$X_0(N)$	$= Y_0(N) \cup (\Gamma_0(N) \setminus \mathbb{P}^1(\mathbb{Q}))$ the compactified modular curve
$\psi_{\mathrm{KM}}(\lambda, \tau, z)$	The Millson theta kernel, 43
$Y_0(N)$	$= \Gamma_0(N) \setminus \mathbb{H}$, the modular curve
$\zeta^*(s)$	The completed Riemann Zeta function, 75
$\mathbb{Z}$	The ring of integers
$\zeta(\Lambda; z)$	The Weierstrass ζ-function, 91
z	$z = x + iy \in \mathbb{H}$
$\widetilde{Z}_{\Delta,r}(m,h)$	A twisted Heegner divisor, 49
$Z(\lambda)$	A Heegner point, 21
$Z(m,h)$	A Heegner divisor, 21
$Z_{\Delta,r}(m,h)$	A twisted Heegner divisor, 49

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