CONGRUENCES FOR RAMANUJAN'S $\omega(q)$

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Abstract. Recently, Bruinier and Ono investigated the arithmetic of the coefficients of Ramanujan's mock theta function $\omega(q)$. In [4] they obtained congruences with respect to the modulus 512. Here we show that $\omega(q)$ modulo 5 is dictated by an elliptic curve.

1. Introduction and statement of results

In a recent paper, Bruinier and Ono obtained congruences for the coefficients of Ramanujan's mock theta function $\omega(q)$ modulo 512. They used the theory of generalized Borcherds products which they developed in [3]. Although their paper concerned derivatives of L-functions, it turns out that their results have interesting implications for partitions and q-series.

Here we illustrate further implications of their results. We consider Ramanujan's mock theta function

$$\omega(q) = \sum_{n=0}^{\infty} a_\omega(n) q^n := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + \cdots \ ,$$

where we use the following notation

$$(a;q)_n := (1-a)(1-aq)\left(1-aq^2\right)\cdots\left(1-aq^{n-1}\right).$$

The following identity

$$q\omega(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q;q^2)_{n+1}},$$

which was obtained by Fine [5], reveals that $q\omega(q)$ is the generating function for a simple partition function. The coefficient $a_\omega(n)$ denotes the number of partitions of $n-1$ whose summands, apart from one of maximal size, form pairs of consecutive non-negative integers.[1]

Example. The partitions of 5 are:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

[1]The interested reader should see Andrew's recent paper [1] for more on the combinatorial interpretation of $\omega(q)$.

Of these partitions, six correspond to partitions whose summands, apart from one of the largest summand, occur in pairs of consecutive integers:

$$5,\, 4+(1+0)\,,\, 3+(1+0)+(1+0)\,,\, 2+(2+1)\,,$$

$$2+(1+0)+(1+0)+(1+0)\,,\, 1+(1+0)+(1+0)+(1+0)+(1+0)\,.$$

And this corresponds to our observation that $a_\omega(4) = 6$.

In this note we investigate the arithmetic properties of the partition function $a_\omega(n)$ with respect to the modulus 5. We shall relate this function to the newform associated to a conductor 30 elliptic curve. Then we will use the theory of Galois representations to obtain density results on the distribution of $a_\omega(n)$ modulo 5.

To this end we define a divisor function using the coefficients $a_\omega(n)$. This function looks "strange" at a first glance, but in fact it arises naturally from one of the generalized Borcherds products associated to $\omega(q)$ (see [3] for a detailed description).

In the following let $\left(\frac{\bullet}{n}\right)$ denote the usual Legendre-symbol and let $\chi(m) := \left(\frac{-8}{m}\right)$ denote the classical Jacobi-symbol character. We define $\hat{\sigma}_\omega(n)$ by

$$\hat{\sigma}_\omega(n) := \sum_{1 \le d \mid n} \left(\frac{d}{3}\right) \chi(n/d)\, d \cdot a_\omega\left(\frac{2d^2-2}{3}\right),$$

and we consider the generating function

$$L_\omega(q) := \sum_{n=1}^{\infty} \hat{\sigma}_\omega(n) q^n = q - 6q^2 + q^3 + 116q^4 - 506q^5 - 6q^6 + \cdots.$$

In [4] Bruinier and Ono showed that the q-series $L_\omega(q)$ is the Fourier expansion of a weight 2 meromorphic modular form on $\Gamma_0(6)$, where here and throughout this note we let $q := e^{2\pi i z}$. We show that this series is interesting geometrically when reduced modulo 5.

Theorem 1.1. *Let $f \in \mathcal{S}_2(\Gamma_0(30))$ be the newform associated to the elliptic curve $E : y^2 + xy + y = x^3 + x + 2$ with Fourier expansion $f(q) = \sum_{n=0}^{\infty} a(n) q^n$. Then the following are true:*
1) We have that

$$L_\omega(q) \equiv f(q) \pmod{5}.$$

2) In particular, if $p \ge 7$ is prime, then we have that

$$\hat{\sigma}_\omega(p) \equiv -\sum_{x \in \mathbb{F}_p} \left(\frac{x^3 + 1917x + 99198}{p}\right) \pmod{5}.$$

Remark. Let $p \geq 7$ be a prime and let here and in the following $\#E(\mathbb{F}_p)$ denote the number of points modulo p on the elliptic curve E. If $p+1 \equiv \#E(\mathbb{F}_p) \pmod 5$, then we have

$$\epsilon(p) + p\left(\frac{p}{3}\right) a_\omega\left(\frac{2p^2-2}{3}\right) \equiv 0 \pmod 5, \text{ where } \epsilon(p) = \begin{cases} 1 & \text{if } p \equiv 1,\, 3 \pmod 8, \\ -1 & \text{if } p \equiv 5,\, 7 \pmod 8. \end{cases}$$

Example. If $p = 61$, then $61 \equiv \#E(\mathbb{F}_{61}) \pmod 5$ and $61 \equiv 5 \pmod 8$. We have that

$$\begin{aligned} \hat{\sigma}_\omega(61) &= -1 + 61\left(\frac{61}{3}\right) a_\omega\left(\frac{2\cdot 61^2 - 2}{3}\right) \\ &= 524263587645622458257610902116440762430. \end{aligned}$$

And we directly see that

$$\hat{\sigma}_\omega(61) \equiv 0 \pmod 5.$$

Corollary 1.2. *For $p \geq 7$ prime and $g(x) := x^3 + 1917x + 99198$ we have that*

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \begin{cases} -p^3\left(\frac{p}{3}\right)\left(1 + \sum\limits_{x\in\mathbb{F}_p}\left(\frac{g(x)}{p}\right)\right) \pmod 5 & \text{if } p \equiv 1, 3 \pmod 8, \\ p^3\left(\frac{p}{3}\right)\left(1 - \sum\limits_{x\in\mathbb{F}_p}\left(\frac{g(x)}{p}\right)\right) \pmod 5 & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

A naive guess would be that for each residue class modulo 5 the congruence $a_\omega\left(\frac{2p^2-2}{3}\right) \equiv rp^3\left(\frac{p}{3}\right) \pmod 5$ holds for one fifth of the time. But here we show that this is not true.

Corollary 1.3. *If for $0 \leq r \leq 4$ we let $\delta(r)$ denote the Dirichlet density of primes p for which*

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv rp^3\left(\frac{p}{3}\right) \pmod 5,$$

then

$$\delta(r) = \begin{cases} \frac{19}{96} & \text{if } r = 0, \\ \frac{13}{64} & \text{if } r = 1, \\ \frac{19}{96} & \text{if } r = 2, \\ \frac{19}{96} & \text{if } r = 3, \\ \frac{13}{64} & \text{if } r = 4. \end{cases}$$

Remark. Let x be a positive integer and let $0 \leq r \leq 4$. Define

$$\delta_x(r) := \frac{\#\{7 \leq p \leq x : \ a_\omega\left(\frac{2p^2-2}{3}\right) \equiv rp^3\left(\frac{p}{3}\right) \pmod 5\}}{\pi(x)},$$

where $\pi(x)$ denotes the number of primes $p \leq x$. Computing $L_\omega(q)$ up to $150,000$ terms we get

$$\delta_x(r) \approx \begin{cases} 0.1981 & \text{if } r = 0, \\ 0.2026 & \text{if } r = 1, \\ 0.1976 & \text{if } r = 2, \\ 0.1985 & \text{if } r = 3, \\ 0.2033 & \text{if } r = 4. \end{cases}$$

We note that $\frac{13}{64} \approx 0.2031$ and $\frac{19}{96} \approx 0.1979$.

In Section 2 we prove these results using Bruinier's and Ono's results about generalized Borcherds products combined with various arguments from the theory of modular forms, elliptic curves, and Galois representations.

2. Proofs

Our results are based on the fact that $L_\omega(q)$ is essentially the logarithmic derivative of the following generalized Borcherds product

$$\begin{aligned} B_\omega(z) :&= \prod_{m=1}^{\infty} \left(\frac{1 + \sqrt{-2}q^m - q^{2m}}{1 - \sqrt{-2}q^m - q^{2m}} \right)^{-4\left(\frac{m}{3}\right) a_\omega\left(\frac{2m^2-2}{3}\right)} \\ &= 1 - 8\sqrt{-2}q - \left(64 - 24\sqrt{-2}\right) q^2 + \left(384 + 168\sqrt{-2}\right) q^3 \cdots . \end{aligned}$$

This function arises as a twisted generalized Borcherds lift for the fundamental discriminant $\Delta = -8$ (see Section 8.2 of [3] for details).

Proof of Theorem 1.1. In order to prove that $L_\omega(q)$ is the reduction of a holomorphic modular form, we recall the construction of $B_\omega(z)$ given in [3]. Let $j_6^*(z)$ be the usual Hauptmodul for $\Gamma_0^*(6)$, the extension of $\Gamma_0(6)$ by all the Atkin-Lehner involutions, which is given by

$$j_6^*(z) := \left(\frac{\eta(z)\eta(2z)}{\eta(3z)\eta(6z)} \right)^4 + 4 + 3^4 \left(\frac{\eta(3z)\eta(6z)}{\eta(z)\eta(2z)} \right)^2 = q^{-1} + 79q + 352q^2 + 1431q^3 + \cdots ,$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta-function. Let α_1 and α_2 be the Heegner points

$$\alpha_1 := \frac{-2 + \sqrt{-2}}{6} \text{ and } \alpha_2 := \frac{2 + \sqrt{-2}}{6}.$$

It is $j_6^*(\alpha_1) = j_6^*(\alpha_2) = -10$. Thus, it follows that $j_6^*(z) + 10$ is a rational modular function on $X_0(6)$ whose divisor consists of the 4 cusps with multiplicity -1 and the points α_1 and α_2 with multiplicity 2.

Let $E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$ be the standard weight 4 Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, and let

$$\delta(z) := \eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2 = q - 2q^2 - 3q^3 + 4q^4 + \cdots .$$

Using $E_4(z)$ and $\delta(z)$, we define the weight 4 holomorphic modular form $\phi(z)$ on $\Gamma_0(6)$

$$450\phi(z) := \left(3360 - 1920\sqrt{-2}\right)\delta(z) + \left(1 - 7\sqrt{-2}\right)E_4(z) + \left(4 - 28\sqrt{-2}\right)E_4(2z) + \left(89 + 7\sqrt{-2}\right)E_4(3z) + \left(356 + 28\sqrt{-2}\right)E_4(6z).$$

Then $\phi(z)$ has divisor $4\,(\alpha_1)$. In terms of $\phi(z)$, $j_6^*(z)$ and $\delta(z)$, it turns out that

$$B_\omega(z) = \frac{\phi(z)}{\left(j_6^*(z) + 10\right)\delta(z)}.$$

We now apply the idea of the proof of Theorem 1 of [2]. Since $E_4(e^{\frac{2\pi i}{3}}) = 0$ and $j(e^{\frac{2\pi i}{3}}) = 0$ it is easy to show that

$$\mathcal{E}(z) := \frac{E_4(z)^6\,(j(z) - 8000)^2}{j(z)^2}$$

is a weight 24 holomorphic modular form. Moreover, it satisfies the congruence

$$\mathcal{E}(z) \equiv 1 \pmod 5.$$

Then, following the proof of Theorem 1 in [2], we see that $\mathcal{E}(z) \cdot \Theta\left(B_\omega(z)\right)/B_\omega(z)$ is a holomorphic modular form of weight 26 on $\Gamma_0\,(6)$, where Θ denotes Ramanujan's Theta-operator. We recall Theorem 1 of [4], namely that $-8\sqrt{-2}L_\omega(q) = \Theta\left(B_\omega(z)\right)/B_\omega(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0\,(6)$. Therefore, we have that $L_\omega(q) \pmod{5^k}$, for every positive integer k, is the reduction of a holomorphic modular form.

Now let $f \in \mathcal{S}_2\left(\Gamma_0\,(30)\right)$ be the newform associated to the elliptic curve

$$E : y^2 + xy + y = x^3 + x + 2$$

and let $f(q) = \sum_{n=0}^{\infty} a(n)q^n$ be its Fourier expansion. Then $f(q)$ is given by

$$f(q) := q - q^2 + q^3 + q^4 - q^5 - q^6 - 4q^7 - q^8 + q^9 + q^{10} + q^{12} + \cdots.$$

We apply Sturm's Theorem (see for example Theorem 2.58 of [6]), which says that $L_\omega(q) \equiv f(q) \pmod 5$ if and only if at least 156 terms agree modulo 5. With the help of a simple computer program we check that this is indeed the case.

To prove part 2) of the theorem we observe that an affine model for E is given by

$$E : y^2 = x^3 + 1917x + 99198.$$

The claim follows directly since the coefficients of the associated newform $f(q)$ are given by

$$a(p) = p + 1 - \#E(\mathbb{F}_p) = -\sum_{x \in \mathbb{F}_p} \left(\frac{x^3 + 1917x + 99198}{p}\right).$$

$\square$

Proof of Corollary 1.2. That

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \begin{cases} -p^3\left(\frac{p}{3}\right)\left(1+\sum\limits_{x\in\mathbb{F}_p}\left(\frac{g(x)}{p}\right)\right) \pmod 5 & \text{if } p \equiv 1,3 \pmod 8, \\ p^3\left(\frac{p}{3}\right)\left(1-\sum\limits_{x\in\mathbb{F}_p}\left(\frac{g(x)}{p}\right)\right) \pmod 5 & \text{if } p \equiv 5,7 \pmod 8, \end{cases}$$

where $g(x) := x^3 + 1917x + 99198$ follows in a straightforward manner from Theorem 1, namely that

$$\hat{\sigma}_\omega(p) \equiv -\sum_{x\in\mathbb{F}_p}\left(\frac{x^3+1917x+99198}{p}\right) \pmod 5,$$

and the definition of $\hat{\sigma}_\omega(p)$. □

Proof of Corollary 1.3. In order to prove the density results, we briefly recall the theory of Galois representations for elliptic curves.

In the following let $l \geq 5$ be prime and let $E/\bar{\mathbb{Q}}$ be an elliptic curve. Then we let $\rho_{E,l}$: $\mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\mathbb{Z}_l\right)$ be the Galois representation associated to E. As usual $\tilde{\rho}_{E,l}$: $\mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\mathbb{Z}/l\mathbb{Z}\right)$ denotes its reduction modulo l. This yields a Galois representation on the l-torsion points of $E/\bar{\mathbb{Q}}$. Then we have that

$$\mathrm{Tr}\left(\rho_{E,l}\left(\mathrm{Frob}_p\right)\right) = a(p)$$

and

$$\det\left(\rho_{E,l}\left(\mathrm{Frob}_p\right)\right) = p,$$

where $a(p) = p + 1 - \#E(\mathbb{F}_p)$. In particular, these identities yield congruences when reducing modulo l.

Let E_5 denote the 5-torsion field of E. We claim that the image of E_5 under $\tilde{\rho}_{E,5}$ is $\mathrm{GL}_2\left(\mathbb{Z}/5\mathbb{Z}\right)$. We can find elements in $\mathrm{Gal}\left(E_5/\mathbb{Q}\right)$ with trace congruent to $r = 0$, 1, 2, 3, 4 respectively and we see that the determinant map det: $\mathrm{Gal}\left(E_5/\mathbb{Q}\right) \to \mathbb{F}_p^*$ is surjective. Thus, we can apply Proposition 19 of [7] which then implies the desired result. The density results then follow from investigating the conjugacy classes of $\mathrm{GL}_2\left(\mathbb{Z}/5\mathbb{Z}\right)$ and the Chebotarev Density Theorem. □

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